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# Interpolation Formulas for Functions with Large Gradients in the Boundary Layer and their Application

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**Abstract.** Interpolation of functions on the basis of Lagrange's polynomials is widely used. However in the case when the function has areas of large gradients, application of polynomials of Lagrange leads to essential errors. It is supposed that the function of one variable has the representation as a sum of regular and boundary layer components. It is supposed that derivatives of a regular component are bounded to a certain order, and the boundary layer component is a function, known within a multiplier; its derivatives are not uniformly bounded. A solution of a singularly perturbed boundary value problem has such a representation. Interpolation formulas, which are exact on a boundary layer component, are constructed. Interpolation error estimates, uniform in a boundary layer component and its derivatives are obtained. Application of the constructed interpolation formulas to creation of formulas of the numerical differentiation and integration of such functions is investigated.

**Keywords:** function of one variable, boundary layer component, nonpolynomial interpolation, quadrature formulas, formulas of numerical differentiation, error estimate.

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# Introduction

Lagrange polynomials for interpolation of functions are widely used. However, according to [1], application of Lagrange polynomials for interpolation of functions with large gradients can lead to large interpolation errors. We suppose that the function under interpolation has the representation as the sum of regular and boundary layer components. Derivatives of the regular component are bounded up to some order. The boundary layer component is known to within a multiplier and has large gradients. It is known that the solution of a singularly perturbed boundary value problem has such representation [2]. We construct the interpolation formula which is exact on the boundary layer component. Then we use the constructed interpolation formula for the creation of formulas of numerical differentiation and integration of functions with the boundary layer component.

Through the paper $C$ and $C_j$ denote generic positive constants independent of $\varepsilon$ and mesh size.

## 1. Interpolation formula and its properties

We suppose that function $u(x)$ has a form:

$$u(x) = p(x) + \gamma\Phi(x),\ \ x \in [a, b], \tag{1}$$

where the function $u(x)$ is smooth enough, the boundary layer component $\Phi(x)$ is known, but its derivatives are not uniformly bounded. Regular component $p(x)$ and its derivatives are uniformly bounded up to some order, the constant $\gamma$ is not known.

The solution of the following singular perturbed problem has the representation (1):

$$\varepsilon u''(x) + a_1(x)u'(x) - a_2(x)u(x) = f(x),\quad u(0) = A,\ \ u(1) = B, \tag{2}$$

where $a_1(x) \geq \alpha > 0,\ \ a_2(x) \geq 0,\ \ \varepsilon \in (0, 1]$, functions $a_1(x), a_2(x), f(x)$ are smooth enough, the constant $\alpha$ is separated from zero. According to [2, 3], the solution of problem (2) has exponential boundary layer near point $x = 0$ and can be written in the form (1).

For example, we will define

$$\Phi(x) = \exp(-a_0\varepsilon^{-1}x),\ \ a_0 = a_1(0),\ \ \gamma = -\varepsilon u'(0)/a_0.$$

Then for some $C_0$ $|p'(x)| \leq C_0$, $|\gamma| \leq C_0$. Derivatives of function $\Phi(x)$ are not $\varepsilon$-uniformly bounded.

Let us $\Omega^h$ be an uniform grid of interval $[a, b]$:

$$\Omega^h = \{x_n :\ \ x_n = a + (n-1)h,\ \ x_1 = a,\ \ x_k = b,\ \ n = 1, 2, \dots, k\}.$$

We suppose that the function $u(x)$ with the representation (1) is given at mesh nodes, $u_n = u(x_n)$.

Let $L_n(u, x)$ is Lagrange polynomial for function $u(x)$ with interpolation conditions at nodes $x_1, x_2, \dots, x_n$. Now we will show that application of a polynomial of Lagrange for interpolation of function (1) can lead to considerable errors. We define $u(x) = e^{-x/\varepsilon}$. Then for $\varepsilon \ll h$ $\ L_2(u, h/2) - u(h/2) \approx 1/2$. The accuracy of the interpolation doesn't increase with decrease of a step $h$.

In [4] for the interpolation of function of a form (1) the following interpolation formula was constructed:

$$L_{\Phi,k}(u, x) = L_{k-1}(u, x) + \frac{[x_1, \dots, x_k]u}{[x_1, \dots, x_k]\Phi}\Big[\Phi(x) - L_{k-1}(\Phi, x)\Big], \tag{3}$$

where $[x_1, \dots, x_k]u$ is the divided difference for a function $u(x)$ [5, p. 43].

It is possible to verify that the formula (3) is interpolation at nodes of $x_1$, $x_2$, $\dots, x_k$, which is exact for polynomials of degree of $(k-2)$ and for component $\gamma\Phi(x)$. This formula is correctly defined if $\Phi^{(k-1)}(x) \neq 0,\ \ x \in (a, b)$. In [4] the following lemma was proved.

**Lemma 1.** *Let*

$$M_k(\Phi, x) = \frac{\Phi(x) - L_{k-1}(\Phi, x)}{\Phi(x_k) - L_{k-1}(\Phi, x_k)}. \tag{4}$$

*Then*

$$\Big|L_{\Phi,k}(u, x) - u(x)\Big| \le \max_s |p^{(k-1)}(s)| \Big[|M_k(\Phi, x)| + 1\Big] h^{k-1}. \tag{5}$$

According to estimate (5), the constructed interpolation formula (3) has the error of order $O(h^{k-1})$ uniformly in $\Phi(x)$ if the function $M_k(\Phi, x)$ is bounded. We proved in [6] that $|M_2(\Phi, x)| \le 1$ if $\Phi'(x) \ne 0, x \in (a, b)$.

**Lemma 2.** *Let*

$$\Phi^{(k-1)}(x) > 0,\ \Phi^{(k)}(x) \ge 0 \quad or \quad \Phi^{(k-1)}(x) < 0,\ \Phi^{(k)}(x) \le 0, \quad x \in (a, b). \tag{6}$$

*Then*

$$|M_k(\Phi, x)| \le 1. \tag{7}$$

*Proof.* Consider the first case in conditions (6). According to [5, p. 44]

$$\Phi(x)-L_{k-1}(\Phi, x) = r_{k-1}(x)[x_1, x_2, \dots, x_{k-1}, x]\Phi,\ \ r_{k-1}(x) = (x-x_1)(x-x_2)\cdots(x-x_{k-1}).$$

Therefore, from (4) we obtain

$$M_k(\Phi, x) = \frac{r_{k-1}(x)[x_1, x_2, \dots, x_{k-1}, x]\Phi}{r_{k-1}(x_k)[x_1, x_2, \dots, x_{k-1}, x_k]\Phi}. \tag{8}$$

According to [5] for some $s \in (a, b)$

$$[x_1, x_2, \dots, x_{k-1}, x]\Phi = \Phi^{(k-1)}(s)/(k-1)!. \tag{9}$$

Considering the condition $\Phi^{(k-1)}(x) > 0$, we obtain $z(x) = [x_1, x_2, \dots, x_{k-1}, x]\Phi > 0$. According to [5, p. 79] $z'(x) = [x_1, x_2, \dots, x_{k-1}, x, x]\Phi$. We use the condition $\Phi^{(k)}(x) \ge 0$ and (9) to obtain $z'(x) \ge 0,\ x \in [a, b]$. So, the function $z(x)$ is positive and increasing on the interval $[a, b]$. Using the inequality $|r_{k-1}(x)| \le r_{k-1}(x_k)$ we obtain (7) from (8). The lemma is proved.

Conditions (6) are satisfied in a case of function $\Phi(x) = e^{(x-1)/\varepsilon}$, corresponding to exponential boundary layer [3].

Using the relation [5, p. 45]

$$L_k(u, x) - L_{k-1}(u, x) = r_{k-1}(x)[x_1, x_2, \dots, x_k]u,$$

the interpolation formula (3) can be written in a form

$$L_{\Phi,k}(u, x) = L_k(u, x) + \frac{[x_1, \dots, x_k]u}{[x_1, \dots, x_k]\Phi}\Big[\Phi(x) - L_k(\Phi, x)\Big]. \tag{10}$$

From (10) obviously follows that $L_{\Phi,k}(u, x)$ is interpolation, which is exact for $\Phi(x)$. We will notice that the formula (10) uses Lagrange's polynomials of bigger degree in comparison with the formula (3).

# 2. Formulas of numerical differentiation

Classical difference formulas for approximate calculation of derivatives are based on polynomials of Lagrange. However, in the case of the function with large gradients the interpolation error of this polynomials can be considerable that affects the accuracy of difference formulas. For such functions it is offered to build formulas of numerical differentiation on the basis of the constructed interpolant (3).

Differentiating the interpolant (3), we obtain

$$u^{(j)}(x) \approx L^{(j)}_{\Phi,k}(u,x) = L^{(j)}_{k-1}(u,x) + \frac{[x_1,\dots,x_k]u}{[x_1,\dots,x_k]\Phi}\Big[\Phi^{(j)}(x) - L^{(j)}_{k-1}(\Phi,x)\Big], \; j \ge 0. \quad (11)$$

The formula (11) is exact for function $\Phi(x)$. The analysis of accuracy of the formula (11) is of interest.

We investigate the accuracy of calculation of the first derivative of function $u(x)$, when the difference formula uses two nodes of any grid interval $[x_{n-1}, x_n]$.

The classical formula for a derivative has the form

$$u'(x) \approx L'_2(u,x) = \frac{u_n - u_{n-1}}{h}, \quad x_{n-1} \le x \le x_n. \quad (12)$$

Let $u(x) = e^{-x/\varepsilon}$. Then $\varepsilon|(u_1 - u_0)/h - u'(0)| = e^{-1}$ if $\varepsilon = h$. So, the relative error of formula (12) doesn't decrease at decrease of a step $h$.

Now we consider the formula (11) with $j = 1, k = 2:$

$$u'(x) \approx L'_{\Phi,2}(u,x) = \frac{u_n - u_{n-1}}{\Phi_n - \Phi_{n-1}}\Phi'(x), \quad x_{n-1} \le x \le x_n. \quad (13)$$

**Lemma 3.** *Suppose that the function $u(x)$ has the representation (1) on the interval $[0,1]$, where*

$$|p'(x)| \le C_1, |p''(x)| \le C_1/\varepsilon, \Phi(x) = e^{-a_0 x/\varepsilon} \text{ or } \Phi(x) = e^{a_0(x-1)/\varepsilon}, \; \varepsilon, a_0 > 0.$$

*Then for some constant $C$*

$$\varepsilon|L'_{\Phi,2}(u,x) - u'(x)| \le Ch, \quad x_{n-1} \le x \le x_n. \quad (14)$$

*Proof.* We consider the case $\Phi(x) = e^{-a_0 x/\varepsilon}$. We take into account that the formula (13) is exact for $\gamma\Phi(x)$ and obtain

$$\varepsilon|L'_{\Phi,2}(u,x) - u'(x)| = \varepsilon|L'_{\Phi,2}(p,x) - p'(x)| \le \varepsilon|L'_{\Phi,2}(p,x) - L'_2(p,x)| + \varepsilon|L'_2(p,x) - p'(x)|.$$

For the second module we have

$$\varepsilon|L'_2(p,x) - p'(x)| \le \varepsilon h \max_s |p''(s)| \le C_1 h. \quad (15)$$

Now we estimate first module. We use the inequality $|p'(x)| \le C_1$ and obtain

$$\varepsilon|L'_{\Phi,2}(p,x) - L'_2(p,x)| \le C_1 h|F(x)|, \quad (16)$$

where

$$F(x) = -a_0 e^{-a_0 x/\varepsilon} \Big/ \Big[ e^{-a_0 x_n/\varepsilon} - e^{-a_0 x_{n-1}/\varepsilon} \Big] - \frac{\varepsilon}{h}.$$

It is obvious that function $F(x)$ decreases on the interval $[x_{n-1}, x_n]$. We will show that this function is uniformly bounded at the ends of the interval. Let $\tau = a_0 h/\varepsilon$. Then

$$F(x_{n-1}) = \frac{\varepsilon}{h} \Big[ \frac{\tau}{1 - e^{-\tau}} - 1 \Big], \quad 0 < \tau < \infty.$$

Considering separately cases $\tau \geq 1$ and $\tau < 1$, we prove that $|F(x_{n-1})| \leq 2a_0$.

The similar estimate is correct for $|F(x_n)|$. Therefore, $|F(x)| \leq 2a_0$. We obtain (14) from (15), (16). The lemma is proved.

Thus, in the case of exponential boundary layer the advantage in accuracy of the offered formula (13) over classical formula (12) is proved.

## 3. Construction of the quadrature formula

Now we consider a question of creation of the quadrature formula for the integral

$$I(u) = \int_a^b u(x)\, dx \tag{17}$$

in the case of the function $u(x)$, having the form (1).

We show that application of formulas of Newton-Cotes can lead to the essential errors. For this purpose we consider the composite Simpson's formula

$$S_3^c(u) = \frac{h}{3} \sum_{n=1,2}^{N-1} \Big( u_{n-1} + 4u_n + u_{n+1} \Big), \quad x_0 = a, x_N = b, Nh = b - a, \;\; a = 0, b = 1.$$

It is known that the composite formula of Simpson has the error about $O(h^4)$ if the derivative $u^{(4)}(x)$ is uniformly bounded.

We consider function $u(x) = \exp(-\varepsilon^{-1}x)$. Derivatives of this function are large for small $\varepsilon$. We write the error of Simpson's formula for the interval $[0, 2h]$

$$\Delta = \int_0^{2h} \exp(-\varepsilon^{-1}x)\, dx - \frac{h}{3}(1 + 4\exp(-\varepsilon^{-1}h) + \exp(-2\varepsilon^{-1}h)).$$

It follows that $\Delta = O(h^5)$ if $\varepsilon \approx 1$ and $\Delta = O(h)$ if $\varepsilon \leq h$. So, in the presence of a boundary layer component the error of Simpson's formula can increase to the quantity about $O(h)$.

In [7] is offered to build quadrature formulas for functions of the form (1) which are exact on the boundary layer component $\Phi(x)$. For this purpose the function under the integral was replaced by interpolant (3), as a result the quadrature formula was constructed. Quadrature formulas with two and three nodes have been constructed. In [8, 9] has been similarly constructed quadrature formulas with four and five nodes. In these works it is proved that the constructed composite formulas have the error of order

$O(h^{k-1})$, where $k$ is the number of nodes of the quadrature formula. There was only restriction $\Phi^{(k-1)}(x) \neq 0,\ x \in (a, b)$.

In [10] the quadrature formula with $k$ nodes, which is exact on a boundary layer component, was constructed.

Substitution of the interpolant (10) into integral (17) leads to a quadrature formula:

$$S_{\Phi,k}(u) = S_k(u) + \frac{[x_1, x_2, \dots, x_k]u}{[x_1, x_2, \dots, x_k]\Phi} \Big[ \int_a^b \Phi(x)\, dx - S_k(\Phi) \Big], \quad S_k(u) = \int_a^b L_k(u, x)\, dx, \quad (18)$$

where $S_k(u)$ is closed Newton-Cotes formula with $k$ nodes. It is obvious that the quadrature formula (18) is exact for the function $\Phi(x)$.

In [10] the estimate of the error of formula (18) is proved.

**Lemma 4.** *Suppose that the function $u(x)$ has the form (1) and the derivative $p^{(k-1)}(x)$ is uniformly bounded on $[a, b]$,*

$$\Phi^{(k-1)}(x) > 0,\ x \in (a, b), \quad \bar{S}_{k-1}(\Phi) \le I(\Phi) \le S_k(\Phi)$$

*or*

$$\Phi^{(k-1)}(x) < 0,\ x \in (a, b), \quad S_k(\Phi) \le I(\Phi) \le \bar{S}_{k-1}(\Phi).$$

*Then*

$$|S_{\Phi,k}(u) - I(u)| \le \frac{2}{(k-1)^{k-1}} \max_s |p^{(k-1)}(s)|(b-a)^k. \quad (19)$$

The error estimate (19) does not depend on the boundary layer component $\Phi(x)$ and its derivatives. The composite quadrature formula, based on a formula (18), has the error of order $O(h^{k-1})$.

As it was told above, in the most usable cases $2 \le k \le 5$ the estimate (19) is fullfiled under one condition $\Phi^{(k-1)}(x) \neq 0,\ a < x < b$.

If at creation of a quadrature formula to use the interpolation formula (3), then we will receive simpler quadrature formula. In particular, at odd $k$ Newton-Cotes's formula in (18) changes on a quadrature formula of open type with smaller number of nodes.

## 4. Numerical results

Consider the function $u(x) = \cos\frac{\pi x}{2} + e^{-\varepsilon^{-1}(x+x^2/2)},\ x \in [0, 1],\ \varepsilon > 0$.

Table 1 contains the maximum error of piecewise polynomial interpolation in the case $k = 4$, computed at the midpoints between the nodes. At small values of $\varepsilon$ the accuracy does not increase with decreasing of step $h$.

Table. 2 contains the interpolation error and the computed order of accuracy of the interpolation (3) with $k = 4$, which is used in subintervals of the length $3h$. At small $\varepsilon$ the order of accuracy becomes third, which corresponds to the estimate (5).

Table 1. The error of piecewise-polynomial interpolation with $k = 4$

| $\varepsilon$ | $N$ | | | | | |
|---|---|---|---|---|---|---|
| | 24 | 48 | 96 | 192 | 384 | 768 |
| 1 | $4.43e-7$ | $2.89e-8$ | $1.84e-9$ | $1.16e-10$ | $7.31e-12$ | $4.58e-13$ |
| $10^{-1}$ | $4.04e-4$ | $2.85e-5$ | $1.88e-6$ | $1.21e-7$ | $7.64e-9$ | $4.80e-10$ |
| $10^{-2}$ | $2.03e-1$ | $7.14e-2$ | $1.28e-2$ | $1.44e-3$ | $1.23e-4$ | $8.99e-6$ |
| $10^{-3}$ | $3.12e-1$ | $3.12e-1$ | $3.07e-1$ | $2.44e-1$ | $1.08e-1$ | $2.41e-2$ |
| $10^{-4}$ | $3.12e-1$ | $3.12e-1$ | $3.12e-1$ | $3.12e-1$ | $3.12e-1$ | $3.11e-1$ |
| $10^{-5}$ | $3.12e-1$ | $3.12e-1$ | $3.12e-1$ | $3.12e-1$ | $3.12e-1$ | $3.12e-1$ |

Table 2. The error and the order of accuracy of interpolation formula (3) with $k = 4$

| $\varepsilon$ | $N$ | | | | | |
|---|---|---|---|---|---|---|
| | 24 | 48 | 96 | 192 | 384 | 768 |
| 1 | $1.20e-5$ | $7.55e-7$ | $4.71e-8$ | $2.94e-9$ | $1.84e-10$ | $1.15e-11$ |
| | 3.99 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $10^{-1}$ | $4.12e-5$ | $2.50e-6$ | $1.52e-7$ | $9.44e-9$ | $5.87e-10$ | $3.66e-11$ |
| | 4.04 | 4.04 | 4.01 | 4.01 | 4.00 | 4.01 |
| $10^{-2}$ | $4.68e-4$ | $2.99e-5$ | $1.70e-6$ | $9.81e-8$ | $5.86e-9$ | $3.57e-10$ |
| | 3.97 | 4.14 | 4.12 | 4.07 | 4.04 | 4.01 |
| $10^{-3}$ | $6.89e-4$ | $8.72e-5$ | $1.08e-5$ | $1.08e-6$ | $7.46e-8$ | $4.28e-9$ |
| | 2.98 | 3.01 | 3.32 | 3.86 | 4.12 | 4.12 |
| $10^{-4}$ | $6.89e-4$ | $8.72e-5$ | $1.09e-5$ | $1.37e-6$ | $1.71e-7$ | $2.13e-8$ |
| | 2.98 | 3.00 | 2.99 | 3.00 | 3.01 | 3.17 |
| $10^{-5}$ | $6.89e-4$ | $8.72e-5$ | $1.09e-5$ | $1.37e-6$ | $1.71e-7$ | $2.14e-8$ |
| | 2.98 | 3.00 | 2.99 | 3.00 | 3.01 | 3.00 |

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**Аннотация.** Интерполяция функций на основе многочленов Лагранжа получила широкое применение. Однако в случае, когда интерполируемая функция имеет области больших градиентов, применение многочленов Лагранжа приводит к существенным погрешностям. В работе предполагается, что интерполируемая функция одной переменной представима в виде суммы регулярной и погранслойной составляющих. Предполагается, что производные регулярной составляющей до определенного порядка ограничены, а погранслойная составляющая является функцией общего вида, известная с точностью до множителя, ее производные не являются равномерно ограниченными. Такое представление имеет решение сингулярно возмущенной краевой задачи. Строятся интерполяционные формулы, точные на погранслойной составляющей, получены оценки погрешности интерполяции, равномерные по погранслойной составляющей и ее производным. Исследовано применение построенных интерполяционных формул к построению формул численного дифференцирования и интегрирования функций рассматриваемого вида.

**Ключевые слова:** функция одной переменной, погранслойная составляющая, неполиномиальная интерполяция, квадратурные формулы, формулы численного дифференцирования

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