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SPLINE INTERPOLATION OF FUNCTIONS WITH A BOUNDARY LAYER COMPONENT

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Abstract. This paper is concerned with the spline interpolation of functions with a boundary layer component. It is shown that polynomial formulas of spline interpolation for such functions lead to significant errors. We proposed nonpolynomial spline interpolation formulas, fitted to a boundary layer component. Nonsmooth and smooth on whole interval interpolants are constructed. The accuracy of constructed formulas is estimated. Numerical results are discussed.

Key words. Function, boundary layer, large gradients, nonpolynomial spline interpolation, uniform accuracy.

1. Introduction

Methods of the spline interpolation for functions with bounded derivatives are well known, see e.g. [1], [15]. But when we apply polynomial interpolation methods to functions with large gradients, it leads to significant errors. In this article we consider an interpolation problem for functions with a boundary layer component. We construct interpolation formulas fitted to a boundary layer component and prove that proposed formulas have uniformly small interpolation error on any uniform mesh. Then we construct difference formulas for a derivative on the base of proposed interpolations.

Let the function $u(x)$ be smooth enough and has the representation

$$u(x) = p(x) + \gamma\Phi(x), \quad x \in [0, 1], \tag{1.1}$$

where $\Phi(x)$ is known function with regions of large gradients and the function $p(x)$ is the regular part of $u(x)$, bounded together with some derivatives, the constant γ is unknown. The representation (1.1) holds for solutions of problems with boundary layers [6].

The purpose of the article is to develop spline interpolation technique on a uniform mesh to functions with a boundary layer component.

Let the function $u(x)$ be given at nodes of the uniform mesh Ω :

$$\Omega = \{x_n : x_n = nh,\ n = 0, 1, \dots, N,\ Nh = 1\},\ \Delta_n = [x_{n-1}, x_n].$$

We denote $u_n = u(x_n),\ n = 0, 1, 2, \dots, N$.

At first, we show that there is a necessity to construct special interpolations for the functions of the form (1.1). We consider the linear interpolation formula

$$u_2(x) = (u_n - u_{n-1})\frac{x - x_n}{h} + u_n,\ x \in \Delta_n. \tag{1.2}$$

It is known [15] that the next estimate holds

$$|u_2(x) - u(x)| \le \frac{h^2}{8} \max_{s\in\Delta_n} |u''(s)|,\ x \in \Delta_n. \tag{1.3}$$

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According to (1.3), we have the second order accuracy if the derivative $u''(x)$ is uniformly bounded. Consider the function $u(x) = \exp(-\varepsilon^{-1}x),\ \varepsilon > 0$. Then the derivative $u''(0) = \varepsilon^{-2}$ is not bounded, if ε tends to zero. Let $\varepsilon = h$, then

$$u_2(h/2) - u(h/2) \approx 0,0774$$

for any step h. The interpolation error can not decrease, when $h \to 0$. So, we need interpolation formulas with the property of uniform accuracy for functions of the form (1.1).

To illustrate that there exist functions with the representation (1.1), we consider the problem

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad u(0) = A,\ u(1) = B, \tag{1.4}$$

where

$$a(x) \geq \alpha > 0, b(x) \geq 0,\ \varepsilon > 0,$$

and the functions $a(x), b(x), f(x)$ are smooth enough. According to [5], the solution of problem (1.4) has exponential boundary layer near the point $x = 0$ and the representation (1.1) for $u(x)$ holds with

$$\Phi(x) = \exp(-a_0\varepsilon^{-1}x) \tag{1.5}$$

and

$$\left|p^{(j)}(x)\right| \leq C_0\left[\varepsilon^{1-j}\exp(-\alpha\varepsilon^{-1}x) + 1\right],\ 0 \leq j \leq 4,$$

where $a_0 = a(0),\ \gamma = -\varepsilon u'(0)/a_0,\quad |\gamma| \leq C_1$. Note that

$$\Phi'(x) < 0,\ \Phi''(x) > 0,\ x \in [0,1].$$

Notations. We mean that $C, C_i,\ i \geq 0$, are some positive constants independent of the function $\Phi(x)$, its derivatives and h. In the case of exponential boundary layer it corresponds to the requirement that C, C_i do not depend on the parameters ε and h.

2. Fitted two-point spline interpolation

We will construct the spline interpolant $u_{\Phi,2}(x)$, taking into account the conditions

$$u_{\Phi,2}(x_{n-1}) = u_{n-1}, \quad u_{\Phi,2}(x_n) = u_n \tag{2.1}$$

for each mesh interval Δ_n. We seek $u_{\Phi,2}(x)$ in the form

$$u_{\Phi,2}(x) = M_1 + M_2\Phi(x).$$

Using conditions (2.1), we obtain

$$u_{\Phi,2}(x) = \frac{u_n - u_{n-1}}{\Phi_n - \Phi_{n-1}}[\Phi(x) - \Phi_n] + u_n,\ x \in \Delta_n, \tag{2.2}$$

where $\Phi_n = \Phi(x_n)$. The interpolant (2.2) is exact for the boundary layer component $\Phi(x)$. We assume that the function $\Phi(x)$ is strictly monotone in each interval Δ_n, then the relation (2.2) is correct. The error of interpolation (2.2) was estimated in [13], where next estimates were obtained

$$|u_{\Phi,2}(x) - u(x)| \leq 2\max_{s\in\Delta_n}|p'(s)|\,h,\ x \in \Delta_n, \tag{2.3}$$

$$|u_\Phi(x) - u(x)| \leq \Big[\max_{s\in\Delta_n}|p''(s)| + \max_{s\in\Delta_n}|p''_{\Phi,2}(s)|\Big]\frac{h^2}{8},\ x \in \Delta_n. \tag{2.4}$$

Here $p_{\Phi,2}(x)$ is the interpolant (2.2) for the function $p(x)$.

According to the estimate (2.4), if a function $u(x)$ has not large gradients on interval Δ_n, then the interpolation formula (2.2) has second order of accuracy. According to (2.3), the interpolation (2.2) has first order of accuracy, in spite of large gradients of $\Phi(x)$.

3. Analogue of Hermite spline interpolation

Consider the following interpolation conditions for an interpolant $u_{\Phi,3}(x)$

$$u_{\Phi,3}(x_{n-1}) = u_{n-1},\ u'_{\Phi,3}(x_{n-1}) = u'_{n-1},\ u_{\Phi,3}(x_n) = u_n,\ x \in \Delta_n, \tag{3.1}$$

where $u'_{n-1} = u'(x_{n-1})$. We construct the interpolant in the form

$$u_{\Phi,3}(x) = M_1 + M_2(x - x_{n-1}) + M_3\Phi(x), \quad x \in \Delta_n. \tag{3.2}$$

We fulfil the conditions (3.1) and obtain

$$u_{\Phi,3}(x) = (u_n - u_{n-1} - hu'_{n-1})\frac{\Phi(x) - \Phi_{n-1} - \Phi'_{n-1}(x - x_{n-1})}{\Phi_n - \Phi_{n-1} - h\Phi'_{n-1}} +$$

$$+u_{n-1} + (x - x_{n-1})u'_{n-1},\ x \in \Delta_n. \tag{3.3}$$

The interpolant (3.3) is exact for a function $\Phi(x)$. In a case $\Phi(x) = x^2$, we obtain Hermite spline interpolation from (3.3)

$$u_3(x) = (u_n - u_{n-1} - hu'_{n-1})\frac{(x - x_{n-1})^2}{h^2} + u_{n-1} + (x - x_{n-1})u'_{n-1},\ x \in \Delta_n. \tag{3.4}$$

The Hermite spline interpolation (3.4) leads to large errors if the function $u(x)$ has a boundary layer component. Now we estimate the accuracy of the interpolation (3.3).

Lemma 3.1. *Let $\Phi''(x) > 0$ or $\Phi''(x) < 0,\ x \in (x_{n-1}, x_n)$. Then,*

$$|u_{\Phi,3}(x) - u(x)| \le \max_{s\in\Delta_n} |p''(s)|h^2, \quad x \in \Delta_n. \tag{3.5}$$

Proof. Let $R_\Phi(u, x) = u_{\Phi,3}(x) - u(x)$. The interpolation $u_{\Phi,3}(x)$ is exact for the function $\Phi(x)$, therefore $R_\Phi(u, x) = R_\Phi(p, x)$. We use (3.3) and obtain

$$R_\Phi(u, x) = (p_n - p_{n-1} - hp'_{n-1})G_n(x) + p_{n-1} + (x - x_{n-1})p'_{n-1} - p(x), \tag{3.6}$$

where

$$G_n(x) = \frac{\Phi(x) - \Phi_{n-1} - \Phi'_{n-1}(x - x_{n-1})}{\Phi_n - \Phi_{n-1} - h\Phi'_{n-1}}.$$

Consider the case $\Phi''(x) > 0$. The function $\Psi(x) = \Phi(x) - \Phi_{n-1} - \Phi'_{n-1}(x - x_{n-1})$ is nonnegative and increasing, consequently,

$$0 \le G_n(x) \le 1, \quad x \in \Delta_n. \tag{3.7}$$

The case $\Phi''(x) < 0$ is analogous to the previous one. Taking into account (3.7), we get the estimate (3.5) from (3.6). $\diamondsuit$

In the case when the function $p''(x)$ is not uniformly bounded, we can use the estimate that follows from (3.6)

$$\Big|u_{\Phi,3}(x) - u(x)\Big| \le 2\int_{x_{n-1}}^{x_n}\int_{x_{n-1}}^{s} \Big|p''(r)\Big|\, dr\, ds,\ x \in \Delta_n.$$

According to [14], the next lemma is true.

Lemma 3.2. *Let $u(x) \in C^3[0,1]$. Then*

$$\Big|u_{\Phi,3}(x) - u(x)\Big| \le \frac{2}{81}\Big[\max_{s\in\Delta_n}|u'''(s)| + \max_{s\in\Delta_n}|u'''_{\Phi,3}(s)|\Big]h^3,\ x \in \Delta_n. \tag{3.8}$$

According to (3.8), the interpolation (3.3) has third order of accuracy, if the function $u(x)$ has not large gradients in the interval Δ_n.

If we take into account that interpolation (3.3) is exact for the component $\Phi(x)$, using (3.8), we obtain

$$\Big|u_{\Phi,3}(x) - u(x)\Big| \le \frac{2}{81}\Big[\max_{s\in\Delta_n}|p'''(s)| + \max_{s\in\Delta_n}|p'''_{\Phi,3}(s)|\Big]h^3,$$

where $p_{\Phi,3}(x)$ corresponds to the interpolation (3.3) for a function $p(x)$.

4. Fitted three-point spline interpolation

In the previous section we constructed an analogue of Hermite spline interpolation using $\{u'_n\}$. However often may be a case, when we know only $\{u_n\},\ n = 0,1,\dots,N$.

To avoid the application of values $\{u'_n\}$ without losing of accuracy, we construct fitted interpolation using three mesh nodes. We construct an interpolant $w_{\Phi,3}(x)$ in the form (3.2) on the interval $[x_{n-1}, x_{n+1}]$, taking into account the conditions

$$w_{\Phi,3}(x_{n-1}) = u_{n-1},\ w_{\Phi,3}(x_n) = u_n,\ w_{\Phi,3}(x_{n+1}) = u_{n+1}. \tag{4.1}$$

We find

$$w_{\Phi,3}(x) = u_n + \frac{u_n - u_{n-1}}{h}(x - x_n)+$$

$$+\frac{u_{n+1} - 2u_n + u_{n-1}}{\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}}\Big[\Phi(x) - \Phi_n - \frac{\Phi_n - \Phi_{n-1}}{h}(x - x_n)\Big],\ x_{n-1} \le x \le x_{n+1}. \tag{4.2}$$

We suppose that N is even,

$$\Phi''(x) > 0 \text{ or } \ \Phi''(x) < 0,\ x \in (x_{n-1}, x_{n+1}).$$

Further we'll get estimates of accuracy for the interpolation (4.2).

4.1. Nonuniform accuracy.

Lemma 4.1. *Let the function $u(x)$ has the representation (1.1). Then,*

$$\Big|w_{\Phi,3}(x) - u(x)\Big| \le \frac{1}{9\sqrt{3}}\Big[\max_s|w'''_{\Phi,3}(s)| + \max_s|u'''(s)|\Big]h^3,\ x, s \in [x_{n-1}, x_{n+1}]. \tag{4.3}$$

Proof. We construct a polynomial interpolant $w_3(x)$ under conditions (4.1)

$$w_3(x) = u_n + \frac{u_n - u_{n-1}}{h}(x - x_n) + \frac{u_{n-1} - 2u_n + u_{n+1}}{2h^2}(x - x_n)(x - x_{n-1}). \tag{4.4}$$

To estimate $|u(x) - w_3(x)|$, we apply the approach [2] and define

$$R(t) = u(t) - w_3(t) - M(t - x_{n-1})(t - x_n)(t - x_{n+1}).$$

Take any $x \in (x_{n-1}, x_{n+1}),\ x \ne x_n$. There is a constant M such that $R(x) = 0$. Then,

$$R(x_{n-1}) = 0,\ R(x) = 0,\ R(x_n) = 0,\ R(x_{n+1}) = 0.$$

We apply the Lagrange theorem three times and obtain that there exists $s_1 \in (x_{n-1}, x_{n+1})$ such that $R'''(s_1) = 0$. Hence, $M = u'''(s_1)/6$. We have

$$u(x) - w_3(x) = \frac{1}{6}u'''(s_1)(x - x_{n-1})(x - x_n)(x - x_{n+1}),\quad x, s_1 \in [x_{n-1}, x_{n+1}].$$

Similarly for some $s_2 \in [x_{n-1}, x_{n+1}]$
$$w_{\Phi,3}(x) - w_3(x) = \frac{1}{6} u'''_{\Phi,3}(s_2)(x - x_{n-1})(x - x_n)(x - x_{n+1}).$$
Hence,
$$|w_{\Phi,3}(x) - u(x)| \le \frac{1}{6}\Big[\max_s |w'''_{\Phi,3}(s) + \max_s |u'''(s)|\Big] \max_x |(x - x_{n-1})(x - x_n)(x - x_{n+1})|.$$
The function
$$g(x) = (x - x_{n-1})(x - x_n)(x - x_{n+1})$$
has extremums at the points $x_{1,2} = x_n \pm h/\sqrt{3}$. Using extremal values of the function $g(x)$ we obtain (4.3). Lemma is proved. $\diamondsuit$

Thus, the interpolation formula (4.2) has third order of accuracy if the function $u(x)$ has bounded derivatives.

4.2. Uniform accuracy. Now we obtain an uniform estimate of the interpolation error for formula (4.2). We investigate the interpolation error in any interval $[x_{n-1}, x_{n+1}]$, $n = 1, 3, \dots, N-1$. Below we will show that under some conditions the following estimate of uniform accuracy

$$|w_{\Phi,3}(x) - u(x)| \le C \max |p''(s)| h^2, \quad x, s \in [x_{n-1}, x_{n+1}] \tag{4.5}$$

is correct. We take into account that the interpolation (4.2) is exact for the function $\Phi(x)$ and obtain

$$w_{\Phi,3}(x) - u(x) = p_{\Phi,3}(x) - p(x) = \Big(p_{\Phi,3}(x) - p_3(x)\Big) + \Big(p_3(x) - p(x)\Big), \tag{4.6}$$

where $p_3(x)$ corresponds to the polynomial interpolation (4.4) for a function $p(x)$. We will estimate expressions in brackets (4.6).

We have
$$p_{\Phi,3}(x) - p_3(x) = \frac{p_{n+1} - 2p_n + p_{n-1}}{2h^2}\Big[2h^2 G(x,h) - (x - x_{n-1})(x - x_n)\Big],$$
$$x_{n-1} \le x \le x_{n+1}, \tag{4.7}$$
where
$$G(x,h) = \frac{\Phi(x) - \Phi_n - (\Phi_n - \Phi_{n-1})(x - x_n)/h}{\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}}. \tag{4.8}$$
It is obviously that
$$G(x_{n-1}, h) = 0, \quad G(x_n, h) = 0, \quad G(x_{n+1}, h) = 1.$$

Let $x \in [x_n, x_{n+1}]$. We shall prove that
$$|G(x,h)| \le 1, \quad x \in [x_n, x_{n+1}]. \tag{4.9}$$
The function $G(x,h)$ increases on x, when $x \in [x_n, x_{n+1}]$. Hence, $0 \le G(x,h) \le 1$ and the inequality (4.9) is true.

Let $x \in [x_{n-1}, x_n]$. The expression (4.8) has the following geometrical interpretation: the modulus of the numerator corresponds to the distance between the function $\Phi(x)$ and its linear interpolant at point $x \in [x_{n-1}, x_n]$, and the denominator corresponds to the distance between the same functions at the point x_{n+1}.

Using Taylor's expansion we obtain
$$G(x,h) = (x - x_{n-1})(x - x_n)\frac{\Phi''_n + (x - x_{n+1})\Phi^{(3)}(s_1)/3}{2h^2\Phi''_n + h^4\Phi^{(4)}(s_2)/6} \tag{4.10}$$
for some $s_1, s_2 \in (x_{n-1}, x_{n+1})$.

At first, consider a case when the function $\Phi(x)$ has bounded derivatives.

Let $|\Phi^{(j)}(x)| \le C$, $j = 2,3,4$. Using (4.10), we obtain $|G(x,h)| \le 1/8 + C_1 h$. It follows that

$$|G(x,h)| \le 1,\ x \in [x_{n-1}, x_n]. \tag{4.11}$$

Then, consider a case when the function $\Phi(x)$ has large derivatives.

Suppose that

$$\Phi''(x) > 0,\ \Phi'(x) < 0,\ x \in (x_{n-1}, x_{n+1}).$$

It is easy to obtain from (4.8)

$$|G(x,h)| \le \frac{\Phi_{n-1} - \Phi_n}{\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}}. \tag{4.12}$$

Note that the replacement of the estimation (4.8) by (4.12) is not strong, if $\Phi(x)$ has large gradients. For some η different from 1 suppose that

$$\Phi_n - \Phi_{n+1} \le \eta\,(\Phi_{n-1} - \Phi_n),\ \eta < 1. \tag{4.13}$$

We use the inequality (4.13) in (4.12) and obtain

$$|G(x,h)| \le \frac{1}{1-\eta}. \tag{4.14}$$

Suppose that

$$\Phi''(x) > 0,\ \Phi'(x) > 0,\ x \in (x_{n-1}, x_{n+1}).$$

In such case

$$|G(x,h)| \le \frac{\Phi_n - \Phi_{n-1}}{\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}}. \tag{4.15}$$

Let us the following inequality

$$\Phi_{n+1} - \Phi_n \ge \eta\,(\Phi_n - \Phi_{n-1}),\ \eta > 1. \tag{4.16}$$

be correct. We use the inequality (4.16) in (4.15) and obtain

$$|G(x,h)| \le \frac{1}{\eta - 1}. \tag{4.17}$$

So, if $x \in [x_{n-1}, x_n]$, we studied the cases, when function $G(x,h)$ is uniformly bounded, according to estimates (4.11), (4.14) and (4.17).

The case $\Phi''(x) < 0$, $x \in (x_{n-1}, x_{n+1})$ is similar.

Consider the example $\Phi(x) = \exp(-a_0\varepsilon^{-1}x)$ based on (1.4). We consider two cases of the relation between ε and h.

Let $a_0\varepsilon^{-1}h < 1$. Using Taylor's expansion of $\Phi(x)$ we obtain $|G(x,h)| \le 1$.

Let $a_0\varepsilon^{-1}h \ge 1$. For such function $\Phi(x)$ the inequality (4.13) becomes the equality with $\eta = \exp(-a_0\varepsilon^{-1}h)$. According to (4.14)

$$|G(x,h)| \le \frac{1}{1 - e^{-1}}.$$

For the given example $|G(x,h)|$ is bounded for any relation between ε and h.

In the case of a right-hand boundary layer function $\Phi(x) = \exp(a_0\varepsilon^{-1}(x-1))$, $x \in [0,1]$ with $a_0\varepsilon^{-1}h \ge 1$ we have $\eta = e$ in (4.17).

So, suppose that $|G(x,h)| \le C_1$. Then we obtain from (4.7)

$$\Big|p_{\phi,3}(x) - p_3(x)\Big| \le C_1 \max_s |p''(s)| h^2, \quad x, s \in [x_{n-1}, x_{n+1}]. \tag{4.18}$$

We need to estimate the second expression in (4.6). We have

$$p_3(x) - p(x) = \frac{1}{h}\int\limits_{x_n}^{x}\int\limits_{x_{n-1}}^{x_n}\int\limits_{s}^{r} p''(\tau)\,d\tau\,dr\,ds +$$

$$+\frac{(x-x_n)(x-x_{n-1})}{2h^2}\Big[\int\limits_{x_{n-1}}^{x_n}(s-x_{n-1})p''(s)\,ds+\int\limits_{x_n}^{x_{n+1}}(x_{n+1}-s))p''(s)\,ds\Big].$$

Estimating each item we obtain

$$\Big|p_3(x)-p(x)\Big|\le\max_s|p''(s)|\frac{h^2}{3}+\max_s|p''(s)|\frac{h^2}{8}=\frac{11}{24}\max_s|p''(s)|h^2. \tag{4.19}$$

Now, we obtain (4.5) from (4.6), (4.18) and (4.19).

5. Analogue of quadratic spline interpolation

In previous sections we constructed interpolants of first and second order of accuracy, uniformly with respect to a boundary layer component. But interpolants were constructed in separated mesh intervals, such interpolants are not continuously differentiable functions within the whole interval $[0,1]$.

Quadratic spline interpolants with continuous first derivative were investigated in [15], [7] and in other works. We shall construct the interpolant $u_\Phi(x)$ with continuous first derivative in $[0,1]$, exact for a boundary layer component $\Phi(x)$.

5.1. Construction of the interpolant. We start from the condition $u_\Phi(x)\in C^1[0,1]$ and define

$$u'_\Phi(x)=M_{n-1}+\Big(M_n-M_{n-1}\Big)\frac{\Phi'(x)-\Phi'_{n-1}}{\Phi'_n-\Phi'_{n-1}},\ x\in\Delta_n, \tag{5.1}$$

where $M_n=u'_\Phi(x_n),\ \Phi'_n=\Phi'(x_n)$. We suppose that

$$\Phi''(x)>0 \text{ or } \Phi''(x)<0,\quad x\in(x_{n-1},x_n),\ n=1,2,\dots,N. \tag{5.2}$$

Under conditions (5.2) the function $\Phi'(x)$ is monotone increasing or monotone decreasing in the interval Δ_n, therefore the formula (5.1) is correct,

$$\Big|\frac{\Phi'(x)-\Phi'_{n-1}}{\Phi'_n-\Phi'_{n-1}}\Big|\le 1,\quad x\in\Delta_n.$$

We integrate (5.1) taking into account the condition $u_\Phi(x_n)=u_n$ and obtain

$$u_\Phi(x)=u_n+M_{n-1}(x-x_n)+\frac{M_n-M_{n-1}}{\Phi'_n-\Phi'_{n-1}}\Big[\Phi(x)-\Phi_n-\Phi'_{n-1}(x-x_n)\Big]. \tag{5.3}$$

We fulfil the interpolation condition $u_\Phi(x_{n-1})=u_{n-1}$ and obtain recurrent relations

$$M_{n-1}+\frac{M_n-M_{n-1}}{\Phi'_n-\Phi'_{n-1}}\Big[\frac{\Phi_n-\Phi_{n-1}}{h}-\Phi'_{n-1}\Big]=\frac{u_n-u_{n-1}}{h},\ n=1,2,\dots,N. \tag{5.4}$$

Define the initial condition for recurrent relations (5.4)

$$M_0=u'(0). \tag{5.5}$$

We use (5.4) in (5.3) and finally obtain

$$u_\Phi(x)=u_n+M_{n-1}(x-x_n)+\Big[u_n-u_{n-1}-M_{n-1}h\Big]\frac{\Phi(x)-\Phi_n-\Phi'_{n-1}(x-x_n)}{\Phi_n-\Phi_{n-1}-\Phi'_{n-1}h},$$

$$x\in[x_{n-1},x_n],\ n=1,2,\dots,N. \tag{5.6}$$

So, we have constructed the interpolant (5.6) with the property $u_\Phi(x)\in C^1[0,1]$, where constants $\{M_n\}$ are defined by relations (5.5), (5.4).

We will prove that interpolation formula (5.6) is exact for the function $\Phi(x)$.

Let $u(x) = \Phi(x)$. It follows from (5.5) that $M_0 = \Phi'_0$. We use that $u_n = \Phi_n$ and by induction obtain from (5.4) that for any n $M_n = \Phi'_n$. Using (5.6), we conclude that $u_\Phi(x) = \Phi(x)$. So, the formula (5.6) is exact for the function $\Phi(x)$.

5.2. Uniform accuracy. Define

$$\Theta_n = \frac{(\Phi_n - \Phi_{n-1})/h - \Phi'_{n-1}}{\Phi'_n - \Phi'_{n-1}}, \; n = 1, 2, \ldots, N. \tag{5.7}$$

Obviously, if the function $\Phi(x)$ is convex or concave inside the interval Δ_n, then $0 < \Theta_n < 1$.

By the following lemma we estimate the accuracy of the interpolation formula (5.6).

Lemma 5.1. *Let conditions (5.2) are fulfilled and for some η separated from* 1,

$$\frac{1-\Theta_n}{\Theta_n} \le \eta < 1, \; n = 1, 2, \ldots, N. \tag{5.8}$$

Then there exists constant C such that

$$|u(x) - u_\Phi(x)| \le C \max_s |p''(s)| h^2. \tag{5.9}$$

Proof. We suppose that $\Phi''(x) > 0, \; x_{n-1} < x < x_n$, the other case in (5.2) can be considered in a similar way.

Let $z_n = M_n - u'_n$. The relation (5.4) is an identity if $u(x) = \Phi(x)$, therefore

$$p'_{\Phi,n-1} + \frac{p'_{\Phi,n} - p'_{\Phi,n-1}}{\Phi'_n - \Phi'_{n-1}} \Big[\frac{\Phi_n - \Phi_{n-1}}{h} - \Phi'_{n-1} \Big] = \frac{p_n - p_{n-1}}{h}, \; n = 1, 2, \ldots, N,$$

$$p'_{\Phi,0} = p'(0). \tag{5.10}$$

It follows from (5.4), (5.5), (5.10) that $z_n = p'_{\Phi,n} - p'_n$. Using (5.10) we obtain

$$\Theta_n z_n + (1 - \Theta_n) z_{n-1} = F_n, \; n = 1, 2, \ldots, N, \quad z_0 = 0, \tag{5.11}$$

where

$$F_n = \frac{p_n - p_{n-1}}{h} - p'_{n-1} - (p'_n - p'_{n-1})\Theta_n. \tag{5.12}$$

From the recurrent relation (5.11) we find

$$z_n = (-1)^n \prod_{j=1}^{n} \frac{1-\Theta_j}{\Theta_j} z_0 + \sum_{j=1}^{n} (-1)^{n-j} \frac{F_j}{\Theta_j} \prod_{i=j+1}^{n} \frac{1-\Theta_i}{\Theta_i}. \tag{5.13}$$

It follows from (5.8) that for any n $\Theta_n > 1/2$. Then we obtain from (5.13)

$$|z_n| \le \eta^n \, |z_0| + 2 \max_n |F_n| \sum_{j=1}^{n} \eta^{n-j} < \eta^n \, |z_0| + \frac{2}{1-\eta} \max_n |F_n|. \tag{5.14}$$

Using (5.12) we get the estimate

$$|F_n| \le \frac{3}{2} \max_s |p''(s)| \, h, \quad s \in [0, 1]. \tag{5.15}$$

Now we use (5.15) in (5.14) to obtain

$$|z_n| \le \eta^n \, |z_0| + 3 \max_s |p''(s)| \, \frac{h}{1-\eta}. \tag{5.16}$$

We take into account the condition $u'_\Phi(0) = u'(0)$ and obtain from (5.16)

$$|M_n - u'_n| \le 3 \max_s |p''(s)| \, \frac{h}{1-\eta}. \tag{5.17}$$

Now we are able to prove the estimate (5.9). We use the condition $\Phi''(x) > 0$ and prove

$$-1 \le \frac{\Phi(x) - \Phi_n - \Phi'_{n-1}(x - x_n)}{\Phi_n - \Phi_{n-1} - \Phi'_{n-1}h} \le 0. \tag{5.18}$$

Introduce an interpolant

$$\tilde{u}_\Phi(x) = u_n + u'_{n-1}(x - x_n) + \Big[u_n - u_{n-1} - u'_{n-1}h\Big]\frac{\Phi(x) - \Phi_n - \Phi'_{n-1}(x - x_n)}{\Phi_n - \Phi_{n-1} - \Phi'_{n-1}h},$$

$$x \in [x_{n-1}, x_n],\ n = 1, 2, \ldots, N. \tag{5.19}$$

We use the interpolation (5.19) which is exact for $\Phi(x)$, take into account (5.18) and obtain

$$|\tilde{u}_\Phi(x) - u(x)| \le \max_s |p''(s)|h^2. \tag{5.20}$$

On the other hand, using (5.17)-(5.19) we obtain that there exists constant C_1 such that

$$|\tilde{u}_\Phi(x) - u_\Phi(x)| \le C_1 \max_s |p''(s)|h^2. \tag{5.21}$$

¿From the estimates (5.20), (5.21) we obtain the estimate (5.9). ♢

Now we analyze the condition (5.8) in the case of the exponential boundary layer component (1.5). Then, Θ_n defined by (5.7) has the form

$$\Theta_n = \frac{1 - (1 - e^{-\tau})/\tau}{1 - e^{-\tau}},\ \tau = \frac{a_0 h}{\varepsilon},\ 0 < \tau < \infty.$$

It is easy to prove that

$$\lim_{\tau \to 0} \Theta_n(\tau) = 1/2, \quad \lim_{\tau \to \infty} \Theta_n(\tau) = 1, \quad \Theta'_n(\tau) > 0.$$

Note that η is close to 1, if $\Theta_n \approx 1/2$. It may be only for $\varepsilon \gg h$. But in this case the classical polynomial spline interpolations may be used.

Now we make restrictions on $\Phi(x)$ to fulfil the inequality $\Theta_n > 1/2$.

Lemma 5.2. *Let us for any* $x \in (x_{n-1}, x_n)$

$$\Phi''(x) > 0,\ \Phi'''(x) < 0 \tag{5.22}$$

or

$$\Phi''(x) < 0,\ \Phi'''(x) > 0. \tag{5.23}$$

Then $\Theta_n > 1/2$, *where* Θ_n *corresponds to (5.7).*

Proof. Let conditions (5.22) hold (the case of conditions (5.23) is similar). Considering (5.7) and (5.22), we rewrite the condition $\Theta_n > 1/2$ in the form

$$\frac{\Phi_n - \Phi_{n-1}}{h} > \frac{1}{2}\big(\Phi'_{n-1} + \Phi'_n\big).$$

We write the last inequality as

$$\int\limits_{x_{n-1}}^{x_n} \Phi'(s)\, ds > \frac{h}{2}\Big(\Phi'_{n-1} + \Phi'_n\Big).$$

This inequality is true, because according to ([3], p. 185) for some $r \in (x_{n-1}, x_n)$

$$\int\limits_{x_{n-1}}^{x_n} \Phi'(s)\, ds = \frac{h}{2}\Big(\Phi'_{n-1} + \Phi'_n\Big) - \frac{h^3}{12}\Phi'''(r)$$

and $\Phi'''(r) < 0$. $\diamondsuit$

Remark. According to conditions (5.22), (5.23) the derivatives $\Phi''(x)$, $\Phi'''(x)$ have different sign. The function $\Phi(x) = \exp(a_0\varepsilon^{-1}(x-1))$ corresponds to the boundary layer near point $x = 1$ and has positive derivatives. In this case $0 < \Theta_n < 1/2$ and we may use a stable recurrent relation, that follows from (5.4):

$$M_{n-1} = \frac{1}{1-\Theta_n}\frac{u_n - u_{n-1}}{h} - \frac{\Theta_n}{1-\Theta_n}M_n, \quad n = N, N-1, \dots, 1, \ M_N = u'(1).$$

5.3. Third order of accuracy. We will prove that the interpolant $u_\Phi(x)$ defined by (5.6) has third order of accuracy, if the function $u(x)$ has not large gradients.

Lemma 5.3. *For the interpolant (5.6) the following estimate holds*

$$|u(x) - u_\Phi(x)| \le \frac{1}{12}\Big[\max_s |u^{(4)}(s)| + \max_s |u_\Phi^{(4)}(s)|\Big]h^3 +$$

$$+\frac{7}{24}\Big[\max_s |u^{'''}(s)| + \max_s |u_\Phi^{'''}(s)|\Big]h^3, \ s, x \in [0,1]. \tag{5.24}$$

Proof. The quadratic spline $w_2(x) \in C^1[0,1]$ follows from (5.4)-(5.6), if $\Phi(x) = x^2$:

$$w_2(x) = u_n + w'_{2,n-1}(x - x_n) + \Big[u_n - u_{n-1} - hw'_{2,n-1}\Big]\frac{(x-x_n)(x-x_{n-1}+h)}{h^2},$$

$$x \in [x_{n-1}, x_n], \ n = 1, 2, \dots, N,$$

where the derivatives $w'_{2,n} = w'_2(x_n)$ are defined by recurrent relations

$$w'_{2,n} = 2\frac{u_n - u_{n-1}}{h} - w'_{2,n-1}, \ n = 1, 2, \dots, N-1, \quad w'_{2,0} = u'(0). \tag{5.25}$$

At first, we estimate $z_n = w'_2(x_n) - u'(x_n)$. Using the relations (5.25) we obtain

$$z_{n-1} + z_n = F_n, \quad F_n = 2\frac{u_n - u_{n-1}}{h} - u'_n - u'_{n-1}, \ 1 \le n \le N, \ z_0 = 0. \tag{5.26}$$

Using (5.26) we can express z_n.

Let n be even. Then

$$z_n = (F_n - F_{n-1}) + (F_{n-2} - F_{n-3}) + \cdots + (F_2 - F_1). \tag{5.27}$$

For each n

$$F_{n+1} - F_n = 2h\Big[\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} - \frac{u'_{n+1} - u'_{n-1}}{2h}\Big].$$

Hence

$$|F_{n+1} - F_n| \le \frac{1}{6}\max_s |u^{(4)}(s)|h^3, \quad s \in [x_{n-1}, x_{n+1}].$$

Using (5.27) we obtain

$$|z_n| \le \frac{1}{12}\max_s |u^{(4)}(s)|h^2, \ s \in [0,1].$$

When n is odd, we obtain

$$z_n = (F_n - F_{n-1}) + (F_{n-2} - F_{n-3}) + \cdots + (F_3 - F_2) + F_1.$$

It follows from (5.26) that

$$|F_1| \le \frac{1}{6}\max_{s\in[0,h]} |u'''(s)|h^2.$$

Hence,

$$(5.28)\quad |w_2'(x_n)-u'(x_n)|\le \frac{1}{12}\max_{s\in[0,1]}|u^{(4)}(s)|h^2+\frac{1}{6}\max_{s\in[0,h]}|u^{'''}(s)|h^2,\ 1\le n\le N.$$

So, for any node x_n the estimate (5.28) holds. For any point x which is not mesh node we need to get similar estimate of accuracy. Let $x\in(x_{n-1},x_n)$.

We introduce the linear interpolation of the function $u'(x)$

$$Int(u',x)=u'_{n-1}+\frac{x-x_{n-1}}{h}(u'_n-u'_{n-1}),\ x\in\Delta_n.$$

Then for any $x\in(x_{n-1},x_n)$

$$(5.29)\quad |w_2'(x)-u'(x)|\le|w_2'(x)-Int(w_2',x)|+|Int(w_2'-u',x)|+|Int(u',x)-u'(x)|.$$

The next estimate is known

$$|Int(u',x)-u'(x)|\le\frac{h^2}{8}\max_{s\in\Delta_n}|u'''(s)|,\ x\in\Delta_n.$$

We use this inequality and (5.28) in (5.29) and otain

$$(5.30)\qquad |w_2'(x)-u'(x)|\le\frac{1}{12}\max_s|u^{(4)}(s)|h^2+\frac{7}{24}\max_s|u^{'''}(s)|h^2,\ s,x\in[0,1].$$

Now we are in position to estimate $|w_2(x)-u(x)|$. For $x\in\Delta_n$ we have

$$w_2(x)-u(x)=(w_2(x)-u(x))-(w_2(x_{n-1})-u(x_{n-1}))=(w_2'(r)-u'(r))(x-x_{n-1}),$$

where $r\in(x_{n-1},x)$. We use (5.30) for $x=r$ and obtain

$$(5.31)\qquad |w_2(x)-u(x)|\le\frac{1}{12}\max_s|u^{(4)}(s)|h^3+\frac{7}{24}\max_s|u^{'''}(s)|h^3,\ s,x\in[0,1].$$

We can consider $w_2(x)$ as a quadratic interpolant for the function $u_\Phi(x)$. Therefore, similarly to (5.31) we have

$$(5.32)\qquad |w_2(x)-u_\Phi(x)|\le\frac{1}{12}\max_s|u_\Phi^{(4)}(s)|h^3+\frac{7}{24}\max_s|u_\Phi^{'''}(s)|h^3,\ s,x\in[0,1].$$

Using (5.31) and (5.32), we obtain (5.24). $\diamondsuit$

To apply the constructed interpolant (5.6) we need $u'(0)$. If we know only $\{u_n\}$ at mesh nodes, we can find $u'(0)$ approximately as derivative of the interpolants (2.2) or (4.2).

6. Calculation of the derivative

6.1. Two-point formula. We will show that an application of the classical difference formulas for a derivative calculation in the case of a function with a boundary layer component leads to significant errors. Consider the formula

$$(6.1)\qquad u'(x)\approx u_2'(x)=\frac{u_n-u_{n-1}}{h},\ x\in\Delta_n,$$

where $u_2(x)$ corresponds to (1.2). Let $u(x)=\exp(-\varepsilon^{-1}x)$, then for $\varepsilon=h$ we have

$$(6.2)\qquad \varepsilon|u_2'(0)-u'(0)|=e^{-1}.$$

We see that the relative error does not decrease when $h\to0$ and $\varepsilon=h$.

If we use the constructed interpolant (2.2), we get an approximate formula

$$(6.3)\qquad u'(x)\approx u'_{\Phi,2}(x)=\frac{u_n-u_{n-1}}{\Phi_n-\Phi_{n-1}}\Phi'(x),\ x\in\Delta_n.$$

The formula (6.3) is exact for $\Phi(x)$. Let us estimate its accuracy. Taking into account the representation (1.1), we obtain

$$u'_{\Phi,2}(x) - u'(x) = \frac{p_n - p_{n-1}}{\Phi_n - \Phi_{n-1}}\Phi'(x) - p'(x).$$

Therefore,

$$u'_{\Phi,2}(x) - u'(x) = \frac{1}{h}\int\limits_{x_{n-1}}^{x_n}\int\limits_{x}^{s}\Big[-\frac{p_n - p_{n-1}}{\Phi_n - \Phi_{n-1}}\Phi''(r) + p''(r)\Big]\,dr\,ds. \tag{6.4}$$

For the classical formula (6.1) we have

$$u'_2(x) - u'(x) = \frac{1}{h}\int\limits_{x_{n-1}}^{x_n}\int\limits_{x}^{s}\Big[\gamma\Phi''(r) + p''(r)\Big]\,dr\,ds. \tag{6.5}$$

Now we can compare the errors (6.4) and (6.5). If $|\Phi_n - \Phi_{n-1}| \gg |p_n - p_{n-1}|$, then the formula (6.3) is more accurate in comparison with (6.1).

Let the function $u(x)$ be a solution of a singular perturbed problem (1.4) with an exponential boundary layer, when $\Phi(x)$ corresponds to (1.5). According to [11] for some C

$$\varepsilon\Big|u'_{\Phi,2}(x) - u'(x)\Big| \le Ch, \quad x \in \Delta_n. \tag{6.6}$$

So, the formula (6.3) has relative error of the order $O(h)$ in contrast to the formula (6.1) with the error of the order $O(1)$.

6.2. Three-point formula. We use the interpolant (4.2) and obtain

$$u'(x) \approx w'_{\Phi,3}(x) = \frac{u_n - u_{n-1}}{h} + \frac{u_{n-1} - 2u_n + u_{n+1}}{\Phi_{n-1} - 2\Phi_n + \Phi_{n+1}}\Big(\Phi'(x) - \frac{\Phi_n - \Phi_{n-1}}{h}\Big), \tag{6.7}$$

where $x \in [x_{n-1}, x_{n+1}]$. For $\Phi(x)$ the constructed formula (6.7) is exact.

We will estimate the accuracy of formula (6.7) using Taylor's series

$$\Big|w'_{\Phi,3}(x) - u'(x)\Big| \le \frac{h^2}{2}\max_s\Big|\frac{\Phi''(s)u'''(s) - u''(s)\Phi'''(s)}{\Phi''(s)}\Big|, \quad x \in [x_{n-1}, x_{n+1}]. \tag{6.8}$$

According to (6.8) the formula (6.7) has the second order accuracy, if the function $u(x)$ has not large gradients and $\Phi''(x)$ is separated from zero.

Now we obtain an estimate for uniform accuracy. For the function $\Phi(x)$ formula (6.7) is exact, therefore,

$$w'_{\Phi,3}(x) - u'(x) = p'_{\Phi,3}(x) - p'(x) = (p'_{\Phi,3}(x) - p'_3(x)) + (p'_3(x) - p'(x)), \tag{6.9}$$

where $p'_{\Phi,3}(x)$ corresponds to the formula (6.7) for the function $p(x)$; $p'_3(x)$ corresponds to the case $\Phi(x) = x^2$ in (6.7),

$$p'_3(x) = \frac{p_n - p_{n-1}}{h} + \frac{p_{n+1} - 2p_n + p_{n-1}}{2h^2}\Big(2x - x_n - x_{n-1}\Big).$$

We use the expressions for $p'_{\Phi,3}(x)$ and $p'_3(x)$ and obtain

$$p'_{\Phi,3}(x) - p'_3(x) = \frac{p_{n-1} - 2p_n + p_{n+1}}{h}\Big(\frac{h\Phi'(x) - \Phi_n + \Phi_{n-1}}{\Phi_{n-1} - 2\Phi_n + \Phi_{n+1}} + \frac{x_n + x_{n-1} - 2x}{2h}\Big),$$

$$x \in [x_{n-1}, x_{n+1}]. \tag{6.10}$$

Let $x = x_n$. Suppose that $\Phi''(x) > 0,\ x \in [x_{n-1}, x_{n+1}]$, a case $\Phi''(x) < 0$ analogous. Note that

$$0 < \frac{h\Phi'(x_n) - \Phi_n + \Phi_{n-1}}{\Phi_{n-1} - 2\Phi_n + \Phi_{n+1}} < 1.$$

Then, using (6.10) we obtain

$$\Big|p'_{\Phi,3}(x_n) - p'_3(x_n)\Big| \le \Big|\frac{p_{n-1} - 2p_n + p_{n+1}}{2h}\Big| \le \frac{1}{2}\int\limits_{x_{n-1}}^{x_{n+1}} |p''(s)|\, ds. \tag{6.11}$$

For the second item in (6.9) we have

$$p'_3(x_n) - p'(x_n) = \frac{1}{2h}\int\limits_{x_{n-1}}^{x_{n+1}}\int\limits_{x_n}^{s} p''(r)\, dr\, ds. \tag{6.12}$$

Now we use the estimates (6.11), (6.12) in (6.9) and obtain

$$\Big|w'_{\Phi,3}(x_n) - u'(x_n)\Big| \le \frac{1}{2}\int\limits_{x_{n-1}}^{x_{n+1}} |p''(s)|\, ds +$$

$$+\frac{1}{2h}\Big[\int\limits_{x_{n-1}}^{x_n}\int\limits_{s}^{x_n} |p''(r)|\, dr\, ds + \int\limits_{x_n}^{x_{n+1}}\int\limits_{x_n}^{s} |p''(r)|\, dr\, ds\Big]. \tag{6.13}$$

In the case of bounded function $p''(x)$ we get from (6.13)

$$\Big|w'_{\Phi,3}(x_n) - u'(x_n)\Big| \le \frac{3}{2}\max_s |p''(s)|h, \quad s \in [x_{n-1}, x_{n+1}]. \tag{6.14}$$

It is interesting that the absolute error has the order $O(h)$, despite the derivative $u'(x_n)$ is not uniformly bounded.

Note that in a case $\Phi(x) = x^2,\ x = x_n$, we deduce from (6.7) the well known formula

$$u'(x_n) \approx \frac{u_{n+1} - u_{n-1}}{2h}. \tag{6.15}$$

Let $x = x_{n-1}$. The expression in brackets in (6.10) is not uniformly bounded, therefore we can estimate only the relative error.

Consider this question in the case of $\Phi(x)$ corresponding to (1.5). Define $\tau = a_0\varepsilon^{-1}h$. Then the expression in brackets in (6.10) has the form

$$\frac{h\Phi'(x_{n-1}) - \Phi_n + \Phi_{n-1}}{\Phi_{n-1} - 2\Phi_n + \Phi_{n+1}} + \frac{1}{2} = -R = -\frac{\tau + 2\exp(-\tau) - 3/2 - (1/2)\exp(-2\tau)}{(1 - \exp(-\tau))^2}.$$

It is easy to prove that $0 < R < 2\tau,\ \ 0 < \tau < \infty$. Using (6.10) we obtain

$$\varepsilon|p'_{\Phi,3}(x_{n-1}) - p'_3(x_{n-1})| \le 2a_0 \max_s |p''(s)|h^2,\ s \in [x_{n-1}, x_{n+1}]. \tag{6.16}$$

Using Taylor's expansion we have

$$|p'_3(x_{n-1}) - p'(x_{n-1})| \le \frac{1}{3}\max_s |p^{(3)}(s)|h^2,\ s \in [x_{n-1}, x_{n+1}]. \tag{6.17}$$

Now we use (6.16), (6.17) in (6.9) and obtain

$$\varepsilon|w'_{\Phi,3}(x_{n-1}) - u'(x_{n-1})| \le 2a_0 \max_s |p''(s)|h^2 + \frac{\varepsilon}{3}\max_s |p^{(3)}(s)|h^2,$$

$$s \in [x_{n-1}, x_{n+1}]. \tag{6.18}$$

If the derivatives $p''(x), p^{(3)}(x)$ are not uniformly bounded, we can obtain more accurate estimation in the integral form instead of (6.18).

7. Application of the constructed interpolants

Interpolants fitted to a boundary layer component can be used in cases, when there is a necessity to interpolate a mesh solution of a singular perturbed problem from nodes of one mesh to nodes of the other mesh. We showed that in the case of the uniform mesh polynomial spline interpolations can lead to significant errors.

When a finite difference scheme is constructed on nonuniform mesh, dense in a boundary layer, there is no necessity to use special interpolations developed in this work. It is known [12] that the linear spline interpolation has the property of uniform with respect to parameter ε accuracy for Shishkin mesh.

In the previous section we got difference formulas for a derivative of a function with boundary layer growth using constructed interpolants.

We found an application of some constructed interpolants to the construction of two-grid methods for nonlinear singular perturbed problems.

In [8], [10] two-grid algorithms for nonlinear ordinary second order singularly perturbed differential equation are constructed. We used Ilyin scheme [4] with the property of an uniform convergence on the uniform mesh. To resolve the difference scheme we investigated Newton and Picard iterative methods. To decrease a number of iterations we preliminarily solved a problem on a coarse grid. Then we need to interpolate found mesh solution to nodes of the initial mesh in order to obtain close initial iteration on a fine mesh. For this purpose we used the interpolation (2.2) in a case of the exponential boundary layer.

In [9] we developed the two-grid method into a system of nonlinear singular perturbed equations.

8. Numerical experiments

We made numerical experiments for verification of accuracy of the constructed interpolants and of the difference formulas for derivative.

8.1. Accuracy of interpolants. Let

$$u(x) = \exp(-\varepsilon^{-1}x) + \frac{1}{x+1}, \ \varepsilon \in (0,1], \ x \in [0,1]. \tag{8.1}$$

For example (8.1) $\Phi(x) = \exp(-\varepsilon^{-1}x)$ and $|p''(x)| \le 2$.

In Table 1 the error

$$\Delta_h = \max_{\varepsilon} \max_{\tilde{x}_n} \Big| u(\tilde{x}_n) - u_{int}(\tilde{x}_n) \Big|$$

for different spline interpolation formulas and h is presented, where

$$\varepsilon \in \{1, 2^{-4}, 2^{-5}, \dots, 2^{-11}\}, \quad \tilde{x}_n = \frac{x_n + x_{n-1}}{2}, \ n = 1, 2, \dots, N, \quad Nh = 1.$$

For the spline interpolation (5.6) we used in (5.5) the approximation

$$u'(0) \approx w'_{\Phi,3}(0) = \frac{u_1 - u_0}{h} + \frac{u_0 - 2u_1 + u_2}{\Phi_0 - 2\Phi_1 + \Phi_2} \Big(\Phi'(0) - \frac{\Phi_1 - \Phi_0}{h} \Big).$$

We define the convergence rate (CR) of the spline interpolation for given value of the parameter ε

$$\delta_h = \max_{\tilde{x}_n} \Big| u(\tilde{x}_n) - u_{int}(\tilde{x}_n) \Big|, \quad CR = \min_h \log_2 \Big(\frac{\delta_h}{\delta_{h/2}} \Big).$$

In Table 2 CR is presented for different values of the parameter ε, depending on the spline interpolation method.

It follows from Tables 1-2 that an application of the polynomial spline interpolation formulas (1.2) and (3.4) leads to significant errors for small values of a parameter ε. Exponential fitted interpolation formulas have the property of an uniform with respect parameter ε convergence. Convergence rate of this methods decreases, when the parameter ε decreases from one to zero, as it was discussed in sections 2-5. Convergence rate decreases from 2 to 1 for two-point fitted interpolation and from 3 to 2 for Hermite, three-point and smooth fitted interpolations.

We will investigate how the accuracy of an interpolation depends on the relation between ε and h on an example of three-point fitted interpolation. In Table 3 the error δ_h of the interpolation (4.2) is presented.

In Table 4 the error δ_h of the smooth spline interpolation (5.6) is presented.

Now we illustrate that it is necessary to take into account $\Phi(x)$, when we approximately define M_0 according to (5.5). In Table 5 the error δ_h of the spline interpolation (5.6) is presented in the case $M_0 = (u_1 - u_0)/h$. We have the essential error for small values of ε.

Consider the next example:

$$u(x) = \exp[-\varepsilon^{-1}(x + x^2/2)] + \cos(x),\ x \in [0, 1]. \tag{8.2}$$

Such function $u(x)$ is the solution of the singularly perturbed problem

$$\varepsilon u'(x) + (1 + x)u(x) = -\varepsilon \sin(x) + (1 + x)\cos(x),\ u(0) = 2.$$

Representation (1.1) for $u(x)$ holds with $\gamma = 1$. The boundary layer component $\Phi(x)$ corresponds to (1.5) with $a_0 = 1$. The derivative $p''(x)$ is not uniformly bounded, $p''(0) = -1/\varepsilon$. Table 6 presents the interpolation error Δ_h for the function (8.2) for various h and different interpolation methods.

8.2. Calculating the derivative. Now we carry out numerical experiments with the difference formulas for the derivative described in section 6.

Let

$$u(x) = \exp(-\varepsilon^{-1}x) + cos(3x),\ \varepsilon \in (0, 1],\ x \in [0, 1].$$

In Table 7 the relative error of formula (6.3)

$$\Delta = \max_{x_n} \Big| u'_{\Phi,2}(x_n) - u'(x_n) \Big| \varepsilon$$

for different values of ε and h is presented. The numerical results confirm the estimate (6.6).

The numerical experiments confirmed also the estimate of accuracy (6.14) for formula (6.7) with $x = x_n$. We obtained in experiments that the classical formula (6.15) leads to large errors, if $\varepsilon \le h$.

Table 1. Maximum error of the interpolation

h	Spline interpolation method					
	linear (1.2)	Hermit (3.4)	fitted two-point (2.2)	fitted Hermit (3.3)	fitted three-point (4.2)	fitted smooth (5.6)
2^{-4}	0.5	31	$2.85e-2$	$8.77e-4$	$2.38e-3$	$1.46e-3$
2^{-5}	0.5	15	$1.49e-2$	$2.26e-4$	$6.58e-4$	$3.66e-4$
2^{-6}	0.5	7.2	$7.63e-3$	$5.58e-5$	$1.73e-4$	$9.15e-5$
2^{-7}	0.5	3.2	$3.86e-3$	$1.31e-5$	$4.45e-5$	$2.29e-5$
2^{-8}	0.482	1.3	$1.87e-3$	$2.75e-6$	$1.08e-5$	$5.45e-6$
2^{-9}	0.374	0.4	$7.41e-4$	$4.79e-7$	$1.99-6$	$1.30e-6$

Table 2. CR of the interpolation

ε	Spline interpolation method					
	linear (1.2)	Hermit (3.4)	fitted two-point (2.2)	fitted Hermit (3.3)	fitted three-point (4.2)	fitted smooth (5.6)
1	1.95	2.94	1.91	2.92	2.84	2.96
2^{-4}	1.67	2.75	1.93	2.91	2.95	3.05
2^{-5}	1.37	2.53	1.86	2.85	2.93	3.04
2^{-6}	0.91	2.18	1.67	2.71	2.75	3.03
2^{-7}	0.37	1.74	1.28	2.46	2.30	2.44
2^{-8}	0.05	1.36	0.98	2.19	1.92	2.06
2^{-9}	0.00	1.15	0.93	2.04	1.85	1.99
2^{-10}	0.00	1.07	0.93	2.07	1.85	1.99

Table 3. The error of three-point fitted interpolation

ε	h				
	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
1	$4.85e-5$	$6.79e-6$	$8.99e-7$	$1.16e-7$	$1.47e-8$
2^{-4}	$3.75e-4$	$4.86e-5$	$6.15e-6$	$7.72e-7$	$9.67e-8$
2^{-5}	$8.55e-4$	$1.12e-4$	$1.40e-5$	$1.75e-6$	$2.17e-7$
2^{-6}	$1.64e-3$	$2.43e-4$	$3.07e-5$	$3.76e-6$	$4.63e-7$
2^{-7}	$2.26e-3$	$4.58e-4$	$6.49e-5$	$8.02e-6$	$9.74e-7$
2^{-8}	$2.37e-3$	$6.26e-4$	$1.21e-4$	$1.68e-5$	$2.05e-6$
2^{-9}	$2.38e-3$	$6.58e-4$	$1.65e-4$	$3.12e-5$	$4.27e-6$
2^{-10}	$2.38e-3$	$6.58e-4$	$1.73e-4$	$4.24e-5$	$7.91e-6$

TABLE 4. The error of the interpolation (5.6) with $M_0 = w'_{\Phi,3}(0)$

ε	h				
	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
1	$1.45E-5$	$1.86E-6$	2.36 E -7	2.96 E -8	3.71 E -9
2^{-4}	$2.67E-4$	$3.22E-5$	3.92 E -6	4.84 E -7	6.01 E -8
2^{-5}	$5.53E-4$	$6.70E-5$	8.06 E -6	9.82 E -7	1.21 E -7
2^{-6}	$1.02E-3$	$1.39E-5$	1.68 E -5	2.02 E -6	2.46 E -7
2^{-7}	$1.39E-3$	$2.57E-4$	3.47 E -5	4.19 E -6	5.04 E -7
2^{-8}	$1.46E-3$	$3.49E-4$	6.42 E -5	8.68 E -6	1.05 E -6
2^{-9}	$1.46E-3$	$3.66E-4$	8.72 E -5	1.60 E -5	2.17 E -6
2^{-10}	$1.46E-3$	$3.66E-4$	9.15 E -5	2.18 E -5	4.01 E -6

TABLE 5. The error of the interpolation (5.6) with $M_0 = (u_1 - u_0)/h$

ε	h				
	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
1	$1.47E-5$	$1.87E-6$	2.36 E -7	2.97 E -8	3.72 E -9
2^{-1}	$1.35E-3$	$3.51E-4$	8.97 E -5	2.27 E -5	5.69 E -6
2^{-2}	$6.42E-3$	$1.71E-3$	4.43 E -4	1.13 E -4	2.84 E -5
2^{-5}	$1.99E-1$	$7.73E-2$	2.44 E -2	6.90 E -3	1.83 E -3
2^{-9}	$5.00E-1$	$5.00E-1$	4.82 E -1	3.74 E -1	2.00 E -1
2^{-11}	$5.00E-1$	$5.00E-1$	5.00 E -1	5.00 E -1	4.82 E -1

TABLE 6. The error of the interpolation for a function (8.2)

h	Spline interpolation method					
	linear (1.2)	Hermit (3.4)	fitted two-point (2.2)	fitted Hermit (3.3)	fitted three-point (4.2)	fitted smooth (5.6)
2^{-4}	0.5	31	$2.60e-2$	$3.11e-3$	$2.49e-3$	$3.11e-3$
2^{-5}	0.5	15	$1.31e-2$	$1.62e-3$	$1.04e-3$	$1.62e-3$
2^{-6}	0.5	7.2	$6.32e-3$	$8.26e-4$	$5.34e-4$	$8.26e-4$
2^{-7}	0.5	3.2	$2.50e-3$	$4.17e-4$	$2.70e-4$	$4.17e-4$
2^{-8}	0.48	1.3	$1.25e-3$	$2.09e-4$	$1.36e-4$	$2.09e-4$
2^{-9}	0.38	0.4	$6.24e-4$	$1.05e-4$	$6.81-5$	$1.06e-4$

TABLE 7. Relative error of the formula (6.3)

ε	h					
	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
1	$2.98e-1$	$1.49e-1$	$7.43e-2$	$3.71e-2$	$1.85e-2$	$9.27e-3$
2^{-4}	$1.11e-1$	$5.169e-2$	$2.48e-2$	$1.22e-2$	$6.02e-3$	$3.00e-3$
2^{-5}	$1.23e-1$	$5.47e-2$	$2.55e-2$	$1.23e-2$	$6.01e-3$	$2.97e-3$
2^{-10}	$1.84e-1$	$9.08e-2$	$4.39e-2$	$2.05e-2$	$9.01e-3$	$3.85e-3$
2^{-11}	$1.59e-1$	$8.13e-2$	$3.95e-2$	$1.90e-2$	$8.83e-3$	$3.87e-3$

9. Conclusion

We investigated the problem of interpolation on an uniform mesh of functions with a boundary layer component. In particular, it may be exponential boundary layer component. It is shown that polynomial spline interpolation leads to significant errors. We proposed formulas that are exact for a boundary layer component. We constructed analogues of liner, Hermite and smooth quadratic spline interpolations. We proved that the constructed interpolants have the property of uniform accuracy (with first order for analogue of liner spline interpolation, with second order for other interpolants). Based on the constructed interpolants, difference formulas for a derivative of the function with the boundary layer component are obtained. We confirmed the theoretical estimates by numerical experiments.

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