On Multi-Product Lot-Sizing and Scheduling with Multi-Machine Technologies

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Abstract

We consider a problem of multi-product lot-sizing and scheduling where each product can be produced by a family of alternative multi-machine technologies. Multi-machine technologies require one or more machine at the same time. A sequence dependent setup time is needed between different technologies. The criterion is to minimize the makespan. Preemptive and non-preemptive versions of the problem are studied. We formulate mixed integer linear programming models based on a continuous time representation for both versions of the problem. Using these models, the polynomially solvable cases of the problem are found. It is proved that the problem without setup times is strongly NP-hard if there is only one product, and each technology occupies at most three machines. Besides that, problem cannot be approximated within a practically relevant factor of the optimum in polynomial time, if P$\neq$NP.

1 Introduction

In practice, many scheduling problems involve tasks, machines and materials such as raw materials, intermediate and final products. Each task may consist in storage, loading/unloading or transformation of one material into another and may be preceded by a sequence-dependent setup time. One of the standard optimization criteria is to minimize the makespan, i.e. the time when the last task is completed.

This paper considers a multi-product lot-sizing and scheduling problem with *multi-machine technologies*, where a multi-machine technology requires more than one machine at the same moment of time, also known as a multi-processor task [3] in parallel computing scheduling. The problem is motivated by the real-life scheduling applications in chemical industry and may be considered as a special case of the problem formulated in [4].

An analysis of computational complexity of lot-sizing and scheduling problems with multi-processor tasks and zero setup times is carried out in [3, 5, 8]. In the present paper we consider a more general case where the non-zero setup times may be required as well. A similar multi-product lot-sizing and scheduling problem with setup times on unrelated parallel machines was studied in [2]. However on one hand, in [2] each technology involved just a single machine, on the other hand the lower bounds on the lot sizes were given.

2 Problem Formulation

Consider a plant producing k different products. Let $V_i > 0$ be the demanded amount of product i, $i = 1, \ldots, k$ and let m be the number of machines available at the plant. For each product i, $i = 1, \ldots, k$, there is at least one technology to produce this product. Let U be the set of all technologies, $d = |U|$, and each technology is characterized by the set of machines it simultaneously occupies $M_u \subseteq \{1, \ldots, m\}$, $u \in U$, and the product i it produces. While the product i is produced by technology u, all machines of the subset M_u are engaged and at any moment each machine of the plant may be engaged in not more than one technology.

Let $U_i \subseteq U$ denote the set of technologies that output product i, $i = 1, \ldots, k$, and $a_u > 0$ is the production rate, i.e. the amount of product i produced by u per unit of time, $u \in U_i$. It is assumed that a feasible schedule may assign to the same product i one or more technologies from U_i, $i = 1, \ldots, k$, i.e. the *migration* is allowed according to the terminology from [8]. For each machine l the setup times from technology u to technology q are denoted by s_{luq}, $s_{luq} > 0$ for all $u,\ q \in K_l$, where $K_l = \{u : \ l \in M_u,\ u \in U\}$ is the set of technologies that use machine l, $l = 1, \ldots, m$.

The problem asks to find for each product i, $i = 1, \ldots, k$, the set of technologies from U_i that will be utilized for production of i, to determine the lot-sizes of production using each of the chosen technologies and to schedule this set of technologies so that the makespan $C_{\max}$ is minimized and the products are produced in demanded volumes $V_1, \ldots, V_k$. The problem is considered in two versions: when preemptions of technologies are allowed (denoted $P|\mathrm{set}_i,\ \mathrm{pmtn},\ \mathrm{s}_{luq}|C_{\max}$) and when the preemptions are not allowed (denoted $P|\mathrm{set}_i,\ \mathrm{s}_{luq}|C_{\max}$).

In practice one often may assume that the setup times satisfy the triangle inequality $s_{luq} + s_{lqp} \geq s_{lup}$, $l = 1, \ldots, m$, $u, q, p \in K_l$. In what follows we denote the special case of preemptive scheduling with the triangle inequality assumption by $P|\mathrm{set}_i,\ \mathrm{pmtn},\ \Delta \mathrm{s}_{luq}|C_{\max}$.

The problems formulated above are strongly NP-hard because in the special case of $m = 1$ the metric shortest Hamilton path reduces to them and this problem is known to be NP-hard in the strong sense [7].

3 Problem Complexity in Case of Zero Setup Times

It was shown in [5, 8] that in case of zero setup times the problems formulated in Section 2 are intractable. These results are obtained using the graph coloring and fractional graph coloring problems. The results from [5, 8, 12] imply that even in the special case when each product has exactly one technology producing it and each machine suits only two technologies, problems $P|\mathrm{set}_i,\ \mathrm{s}_{luq}=0|C_{\max}$ and $P|\mathrm{set}_i,\ \mathrm{pmtn},\ \mathrm{s}_{luq}=0|C_{\max}$ can not be approximated within a factor $k^{1-\varepsilon}$ for any $\varepsilon>0$, if P≠NP.

Here we claim that in the case of single product, when multiple technologies are allowed, the problems formulated in Section 2 are intractable as well:

Proposition 1 *Problems* $P|\mathrm{set}_i,\ \mathrm{s}_{luq}=0|C_{\max}$ *and* $P|\mathrm{set}_i,\ \mathrm{pmtn},\ \mathrm{s}_{luq}=0|C_{\max}$ *are strongly NP-hard even in the special case when the number of products* $k=1$*, and all technologies have equal production rates, however each technology occupies at most 3 machines.*

Besides that, in the case of $k=1$*, the problems* $P|\mathrm{set}_i,\ \mathrm{s}_{luq}=0|C_{\max}$ *and* $P|\mathrm{set}_i,\ \mathrm{pmtn},\ \mathrm{s}_{luq}=0|C_{\max}$ *are not approximable within a factor* $d^{1-\varepsilon}$ *for any* $\varepsilon>0$*, assuming P≠NP.*

4 Mixed Integer Programming Model

Let us define the notion of *event points* analogously to [6]. By event point we will mean a subset of variables in mixed integer programming (MIP) model, which characterize a selection of a certain set of technologies and their starting and completion times. In one event point each machine may be utilized in at most one technology. The set of all event points will be denoted by $N=\{1,\ldots,n_{\max}\}$, where the parameter $n_{\max}$ is chosen sufficiently large on the basis of a-priory estimates or preliminary experiments.

The structure of the schedule is defined by the Boolean variables w_{un} such that $w_{un}=1$ if technology u is executed in event point n, and $w_{un}=0$ otherwise. In case technology u is executed in event point n, the staring time and the completion time of technology u in this event point are given by the real-valued variables T^s_{un} and T^f_{un} accordingly. The variable $C_{\max}$ is equal to the time when the last technology is finished (the makespan).

Define the following notation:
let I be the set of all products, $|I|=k$;
let M be the set of machines, $|M|=m$;
$H=\sum\limits_{i\in I}\max\limits_{u\in U_i}\left\{\frac{V_i}{a_u}\right\}+(k-1)\cdot\max\limits_{l\in M,\ u,q\in K_l}\{s_{luq}\}$ is an upper bound on makespan. The amount of time H is sufficient to produce all the demanded products.

Then the MIP model for $P|\mathrm{set}_i,\ \mathrm{pmtn},\ \mathrm{s}_{luq}|C_{\max}$ problem is as follows:

$$C_{\max} \to \min, \tag{1}$$

$$T^f_{un} \le C_{\max},\ u \in U,\ n \in N, \tag{2}$$

$$\sum_{u \in K_l} w_{un} \le 1,\ l \in M,\ n \in N, \tag{3}$$

$$T^s_{un} \ge T^f_{q\tilde{n}} + s_{lqu} - H \cdot (2 - w_{un} - w_{q\tilde{n}} + \sum_{q' \in K_l} \sum_{\tilde{n} < n' < n} w_{q'n'}), \tag{4}$$

$$l \in M,\ u,\ q \in K_l,\ n,\ \tilde{n} \in N,\ n \ne 1,\ \tilde{n} < n,$$

$$T^f_{un} \ge T^s_{un},\ u \in U,\ n \in N, \tag{5}$$

$$T^f_{un} - T^s_{un} \le w_{un} \cdot \max_{q \in U_i} \left\{ \frac{V_i}{a_q} \right\},\ i \in I,\ u \in U_i,\ n \in N, \tag{6}$$

$$\sum_{n \in N} \sum_{u \in U_i} a_u \cdot (T^f_{un} - T^s_{un}) \ge V_i,\ i \in I, \tag{7}$$

$$T^s_{un} \ge 0,\ u \in U,\ n \in N, \tag{8}$$

$$w_{un} \in \{0, 1\},\ u \in U,\ n \in N. \tag{9}$$

The objective function (1) and inequality (2) define the makespan criterion. Constraint (3) implies that in any event point on machine l at most one technology may be executed. Constraint (4) indicates that the starting time of technology u on machine l should not be less than the completion time of a preceding technology on the same machine, plus the setup time. Constraint (5) guarantees that all technologies may be performed only for non-negative time. If a technology u is not executed in the event point n (i.e. $w_{un} = 0$) then its duration should be zero – this is ensured by inequality (6). Constraint (7) bounds the amount of production according to the demand. Constraints (8) – (9) give the area where the variables are defined.

A MIP model for problem $P|\mathrm{set}_i,\ \mathrm{s}_{luq}|C_{\max}$ may be obtained from (1) – (9) by adding the inequality

$$\sum_{n \in N} w_{un} \le 1,\ u \in U, \tag{10}$$

which ensures each technology is executed without preemptions.

These two models and their modifications for the triangle inequality case are studied experimentally in [10].

5 Polynomially Solvable Cases

In order to find an optimal solution to $P|\mathrm{set}_i,\ \mathrm{s}_{luq}|C_{\max}$ using model (1) – (10), it is sufficient to set $n_{\max} = d$ because the preemptions are not allowed. Denote $\mathcal{P}_{LP}$ the linear programming problem obtained by fixing all Boolean variables (w_{un}) in model (1) – (10). Here and below by *fixing* of the variables we assume assignment of some fixed values to them (which turns these variables into parameters). Problem $\mathcal{P}_{LP}$ with $n_{\max} = d$ involves a polynomially bounded number of variables, which means it is polynomially solvable (see e.g. [9]).

Let τ_{LP} be an upper bound on the time complexity of solving problem $\mathcal{P}_{LP}$. The problem $P|\text{set}_i,\ \text{s}_{luq}|C_{\max}$, where the number of technologies is bounded by a constant from above, we will denote by $P|\text{set}_i,\ \text{s}_{luq}, d = const|C_{\max}$. This problem reduces to $(n_{\max}+1)^d$ problems of $\mathcal{P}_{LP}$ type with $n_{\max} = d$. Therefore the following theorem holds.

Theorem 1 *Problem* $P|\text{set}_i,\ \text{s}_{luq}, d = const|C_{\max}$ *is polynomially solvable within* $O(\tau_{LP} \cdot d^d)$ *time.*

To find an optimal solution to $P|\text{set}_i,\ \text{pmtn},\ \Delta\text{s}_{luq}|C_{\max}$ problem, it suffices to set $n_{\max} = d^m$ in model (1) – (9). Indeed, the number of different sets of technologies that may be executed simultaneously does not exceed $\prod_{l=1}^{m} f_l \leq d^m$, where $f_l = |K_l| + 1$ if $|K_l| < d$, otherwise $f_l = d$. Besides that, there exists an optimal solution to problem $P|\text{set}_i,\ \text{pmtn},\ \Delta\text{s}_{luq}|C_{\max}$ where each of the above mentioned sets of technologies is executed simultaneously at most once. This fact follows by the lot shifting technique which is applicable here since the setup times obey the triangle inequality.

Let $\mathcal{P}'_{LP}$ denote the linear programming problem obtained by fixing all Boolean variables (w_{un}) in MIP model (1) – (9). A problem $\mathcal{P}'_{LP}$ with $n_{\max} = d^m$ and the number of machines bounded above by a constant is polynomially solvable. Let τ'_{LP} denote an upper bound of the time complexity of solving $\mathcal{P}'_{LP}$. The problem $P|\text{set}_i,\ \text{pmtn},\ \Delta\text{s}_{luq}|C_{\max}$, where the numbers of machines and products are bounded by a constant will be denoted by $Pm|\text{set}_i,\ \text{pmtn},\ \Delta\text{s}_{luq}, k = const|C_{\max}$ in what follows. This problem reduces to $2^{dn_{\max}}$ problems of $\mathcal{P}'_{LP}$ type, where $n_{\max} = d^m$. The total number of technologies d does not exceed $k(2^m - 1)$, so the following result holds.

Theorem 2 *Problem* $Pm|\text{set}_i,\ \text{pmtn},\ \Delta\text{s}_{luq}, k = const|C_{\max}$ *is polynomially solvable within* $O(\tau'_{LP} \cdot 2^{(k(2^m-1))^{m+1}})$ *time.*

A number of other polynomially solvable cases of problems $P|\ \text{set}_i,\ \text{s}_{luq}|\ C_{\max}$ and $P|\ \text{set}_i,\ \text{pmtn},\ \text{s}_{luq}|\ C_{\max}$ with zero setup times may be found in [1, 5, 8, 11].

Conclusion

The problem of multi-product lot-sizing and scheduling with multi-machine technologies is studied in preemptive and non-preemptive versions. Non-approximability of the problem is shown and new NP-hard special cases with zero setup times are identified. MIP models are formulated for both versions of the problem using the event-points approach and continuous time representation. New polynomially solvable special cases of the problem are found using the MIP models, under assumption that the number of technologies is bounded by a constant.

Further research appears to be appropriate in extending the obtained results to the version of the problem where technologies may involve several tasks which should be executed sequentially and each task is performed on a number of machines simultaneously.

6 Acknowledgement

This research have been supported by the RFBR Grants 12-01-00122 and 13-01-00862.

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