ON THE QUASIVARIETY GENERATED BY A NON-CYCLIC FREE METABELIAN GROUP

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Abstract. Our main result is a characterization of the finitely generated groups in the quasivariety generated by a non-cyclic free metabelian group from three different points of view: In terms of wreath products, in terms of module theoretic properties of their Fitting subgroups, and in terms of quasi-identities.

1. Introduction

In this paper we study the finitely generated groups in the quasivariety of groups qvar(F) that is generated by a non-cyclic free metabelian group F. We mention at once that qvar(F) does not depend on the rank of the generating free metabelian group since all free metabelian groups of rank greater than one generate the same quasivariety (see Section 2.3). Our main result, Theorem B in Section 7, is a characterization of these groups from different points of view: In terms of wreath products (any finitely generated group in qvar(F) can be embedded into a direct product of wreath products of free abelian groups), in terms of modules (the groups in question are determined by module theoretic properties of their Fitting subgroups), and, finally, in terms of quasi-identities (we exhibit a recursive system of quasi-identities that determines qvar(F)). The motivation for this work came from algebraic geometry over groups, a new concept in group theory that has recently been developed in [1] and [8]. A central role in this new theory is played by algebraic sets over groups and their coordinate groups. By a result of [8], the coordinate groups of algebraic sets over free metabelian groups are precisely the finitely generated groups in the quasivariety qvar(F). The present paper has been written with this application in mind, but we hope that our main result will be of independent interest as a contribution to the theory of quasivarieties.

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In the proof of our main theorem we make use of results of Chapuis [3] on the universal closure $\mathrm{ucl}(F)$, that is the class of all groups satisfying the universal theory of a non-cyclic free metabelian group. In fact, in Section 7 we prove a modification of the main result of [3], Theorem A, which gives a characterization of the finitely generated groups in $\mathrm{ucl}(F)$ in terms similar to those used in Theorem B.

The paper is organized as follows. Notation and some preliminary notions will be introduced in Section 2. In Sections 3 and 4 we introduce and discuss specific classes of rings and modules, so called $\mathcal{A}$-rings and $\mathcal{A}$-modules, and these are then used in Section 5 to define metabelian $\mathcal{A}$-groups, one of the central concepts of this paper. In Section 6 we give the defining system of quasi-identities for $\mathrm{qvar}(F)$, and the final Section 7 is devoted to the proofs of Theorems A and B.

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2. Notation and preliminary notions

2.1. **Groups.** We write A_n for the free abelian group of rank n, and sometimes, when the rank is understood, we use A. By $\mathrm{Pr}(A_n)$ we denote the set of all primitive elements in A_n, and we let $\mathcal{P}(A_n)$ denote the set of all pure (or isolated) subgroups of A_n (which is the same as the set of all direct factors).

If G is a group, we let $\mathrm{Fit}(G)$ denote the Fitting subgroup of G, we write G' for the commutator subgroup of G, and $Z(G)$ for the centre. We use standard notation for conjugates and commutators: $g^h = h^{-1}gh$ and $[g,h] = g^{-1}h^{-1}gh$ for $g, h \in G$. If N is an abelian normal subgroup of G, then N carries the structure of a right $\mathbb{Z}G$-module via conjugation in G, and since N itself acts trivially, this is, in fact, a $\mathbb{Z}(G/N)$-module. In this situation we will use multiplicative notation for the module N with the acting elements of the group ring appearing as exponents. Thus, if $u \in N$, $\alpha = \sum_{i=1}^k n_i\overline{g}_i \in \mathbb{Z}(G/N)$ and $\sum_{i=1}^k n_i g_i \in \mathbb{Z}G$ with $n_i \in \mathbb{Z}$, $g_i \in G$ and $\overline{g_i} = g_iN \in G/N$, then

$$u^\alpha = u^{\sum_{i=1}^k n_i\overline{g}_i} = u^{\sum_{i=1}^k n_i g_i} = (u^{n_1})^{g_1}(u^{n_2})^{g_2}\dots(u^{n_k})^{g_k}.$$

By $W_{r,s}$ we denote the restricted wreath product of two free abelian groups of ranks r and s, respectively: $W_{r,s} = A_r \mathrm{wr} A_s$. For our purposes it will be convenient to regard $W_{r,s}$ as the semidirect product of A_s and a (multiplicatively written) free right $\mathbb{Z}A_s$-module T of rank r (the base group). The elements of $W_{r,s}$ will be written as ordered pairs (a, t) with $a \in A_s$ and $t \in T$. Elementary properties of the wreath products $W_{r,s}$ will be used without special references being given.

An important concept in this paper is the notion of a ρ-group which is due to Chapuis [3].

Definition. A group G is called a ρ-group if the following three conditions hold.
(i) G is a torsion-free metabelian group,
(ii) the Fitting subgroup $\mathrm{Fit}(G)$ is abelian and isolated in G,
(iii) $\mathrm{Fit}(G)$ is a torsion-free module over $\mathbb{Z}(G/\mathrm{Fit}(G))$.

The wreath products $W_{r,s}$ are obvious examples of ρ-groups. Another important notion in this paper is that of commutation-transitive groups.

Definition. A group G is called commutation-transitive (or CT-group) if satisfies the axiom

$$CT: \quad \forall x, y, z((x \neq 1) \wedge [x, y] = 1 \wedge [x, z] = 1 \Longrightarrow [y, z] = 1).$$

In other words, G is commutation-transitive if any two elements $y, z \in G$ which commute with a third element x $(x \neq 1)$ of G, commute with each other.

Finally, we write F for a non-cyclic free metabelian group, and we use F_n when we wish to specify the rank. We will assume that the reader is familiar with basic facts about free metabelian groups. In particular, for F_n we have that $F_n/F_n' \cong A_n$, and that $\mathrm{Fit}(F_n) = F_n'$. Moreover $\mathrm{Fit}(F_n)$ is a free abelian group, and will be regarded as a $\mathbb{Z}A_n$-module via conjugation in F_n. An important tool in studying free metabelian groups is the well-known Magnus embedding

$$\mu : F_n \to W_{n,n}$$

which is given by

$$x_i \mapsto (a_i, t_i) \qquad (i = 1, \ldots, n)$$

where the x_i and a_i denote the the free generators of F_n and A_n, respectively, and the t_i denote the free module-generators of the base group

T of $W_{n,n}$. The restriction of μ to $\mathrm{Fit}(F_n)$ maps this module isomorphically into the free module T. It follows that $\mathrm{Fit}(F_n)$ is a torsion-free $\mathbb{Z}A_n$-module. The embedding μ is originally due to Magnus [6] who used a certain matrix group instead of $W_{n,n}$. Using the Magnus embedding, it is not hard to show some further well-known facts about free metabelian groups. In particular, one gets easily that these groups are linear. For reference purposes we mention the following well-known result about centralizers in F_n, which is a special case of a theorem due to Mal'cev [7]. For the centralizer $C(g)$ of an element $g \in F_n$ with $g \neq 1$ one has $C(g) = F_n'$ if $g \in F_n'$, and if $g \notin F_n'$, then $C(g)$ coincides with the maximal cyclic subgroup of F_n containing g. This implies, in particular, that the centre of F_n is trivial and that F_n satisfies the axiom CT.

2.2. **Rings and modules.** All rings in this paper are factor rings of the integral group rings $R_n = \mathbb{Z}A_n$, and all modules are (right) modules over these rings. In particular, all rings and all finitely generated modules under consideration will be Noetherian. When working in groups, where abelian normal subgroups will be regarded as modules via conjugation (as explained in Section 2.1), modules will be written multiplicatively, but otherwise (here and in Section 4) we shall use the common additive notation.

For a subgroup $B \leq A_n$, we let Δ_B denote the ideal of R_n that is generated by all elements of the form $1 - b$ where $b \in B$. In particular, Δ_{A_n} is the augmentation ideal, that is the kernel of the augmentation map $\varepsilon : \mathbb{Z}A_n \to \mathbb{Z}$. It is well-known that, if B is generated by its elements $b_1, b_2, \ldots$, then the elements $1 - b_1, 1 - b_2, \ldots$ generate Δ_B as an ideal.

Any element $\alpha \in R_n$ has a unique expression as a linear combination

$$\alpha = \sum_{g \in A_n} n_g g \tag{2.1}$$

where $n_g \in \mathbb{Z}$ and only finitely many of the coefficients n_g are non-zero. We define the support of α by setting $\mathrm{supp}(\alpha) = \{g \in G\ ;\ n_g \neq 0\}$, and the length of α by $|\alpha| = |\,\mathrm{supp}(\alpha)|$. Now we introduce the important notion of the *contents* $C(\alpha)$ of α. For α as in (2.1) let $\mathrm{supp}^+(\alpha) = \{g \in G\ ;\ n_g > 0\}$ and $\mathrm{supp}^-(\alpha) = \{h \in G\ ;\ n_h < 0\}$. If $\mathrm{supp}^+(\alpha) = \emptyset$ or $\mathrm{supp}^-(\alpha) = \emptyset$, we set $C(\alpha) = \emptyset$. Otherwise, for any pair of elements $g \in \mathrm{supp}^+(\alpha)$, $h \in \mathrm{supp}^-(\alpha)$, let $b \in \mathrm{Pr}(A_n)$ be the highest possible root of gh^{-1} in A_n, say $gh^{-1} = b^{l_{g,h}}$ for some integer $l_{g,h} \geq 1$. Then $C(\alpha)$ is defined as the set of all such primitive elements b. Thus

$$C(\alpha) = \{b \in \mathrm{Pr}(A_n)\ ;\ b^{l_{g,h}} = gh^{-1} \text{ for some } l_{g,h} \geq 1,\ g \in \mathrm{supp}^+(\alpha),\ h \in \mathrm{supp}^-(\alpha)\}.$$

It is clear that $C(\alpha)$ is a finite set. Note that $C(\alpha) \neq \emptyset$ for all non-zero $\alpha \in \Delta_{A_n}$.

We need some standard material from commutative algebra about primary decomposition of Noetherian modules. All of this can be found, for example, in [2] or [5]. The following notation will be used. Let S be a commutative ring and M a (right) S-module. For $u \in M$ we write $\mathrm{Ann}(u)$ for the annihilator of u,

$$\mathrm{Ann}(u) = \{\alpha \in S\ ;\ u\alpha = 0\},$$

we write $\mathrm{Ass}(M)$ for the prime ideals of S that are associated with M,

$$\mathrm{Ass}(M) = \text{the set of prime ideals associated with } M,$$

if I is an ideal in S, we set

$$M[I] = \{u \in M\ ;\ u\alpha = 0 \text{ for all } \alpha \in I\},$$

and, for ideals $I, J \lhd S$ we set

$$(I : J) = \{\beta \in S\ ;\ \beta\alpha \in I \text{ for all } \alpha \in J\}.$$

2.3. **Quasivarieties and Universal Closures.** We shall work with the standard language L of group theory consisting of a symbol $\cdot$ for multiplication, a symbol $^{-1}$ for inversion, and a symbol 1 for the identity. A universal sentence in L is formula of the form

$$\forall x_1, \ldots, x_n (\bigvee_{j=1}^{s} \bigwedge_{i=1}^{t} (u_{ij}(\overline{x}) = 1 \wedge w_{ij}(\overline{x}) \neq 1))$$

where $u_{ij}(\overline{x})$ and $u_{ij}(\overline{x})$ are group words in the variables $x_1, \ldots, x_n$. The universal theory $\mathrm{Th}_\forall(G)$ of a group G consists of all universal sentences in L which are true on G, and the universal closure $\mathrm{ucl}(G)$ of G consists of all groups H such that all universal sentences from $\mathrm{Th}_\forall(G)$ are also true on H.

A quasi-identity in L is a formula of the form

$$\forall x_1, \ldots, x_n ((\bigwedge_{i=1}^{t} u_i(\overline{x}) = 1) \Rightarrow s(\overline{x}) = 1)$$

where $u_i(\overline{x})$ and $s(\overline{x})$ are group words in the variables $x_1, \ldots, x_n$. A quasivariety of groups is a class groups that can be axiomatized by a set of quasi-identities. In other words, a class of groups is a quasivariety if there exists a set of quasi-identities such that the class consists of all groups satisfying all of these quasi-identities. The quasivariety $\mathrm{qvar}(G)$ generated by a group G is smallest quasivariety containing G.

In this paper we focus on non-cyclic free metabelian groups, their universal closures, and, in particular, the quasivarieties they generate. It is known (see [4]) that $\mathrm{ucl}(F_n) = \mathrm{ucl}(F_m)$ for all $n, m \geq 2$, and hence it makes good sense to speak of the universal closure $\mathrm{ucl}(F)$ of a non-cyclic free metabelian group. As we have mentioned at the very beginning, the situation is similar for quasivarieties.

Lemma 2.1. *The quasivarieties* $\mathrm{qvar}(F_n)$ *and* $\mathrm{qvar}(F_m)$ *coincide for all* $n, m \geq 2$.

Proof. We show that $\mathrm{qvar}(F_n) = \mathrm{qvar}(F_2)$ for all $n > 2$. Since F_2 is a subgroup of F_n, we have obviously that $\mathrm{qvar}(F_2) \subseteq \mathrm{qvar}(F_n)$. On the other hand, since F_n is, for all $n \geq 2$, residually F_2 (see, for example, [9, 36.35]), the inverse inclusion follows immediately from the fact that $\mathrm{qvar}(F_2)$ consists precisely of all groups which are locally residually F_2. The latter holds, in fact, for a large class of groups, so-called equationally Noetherian groups, which includes our F_2 by [1, Theorem B1] because free metabelian groups of finite rank are linear (see Section 2.1). The required fact itself is implicitly contained in [8]. On noting that (in the terminology of [8]) equationally Noetherian groups are q_ω-compact, it follows by combining Theorem B1 and Lemma 7 of [8]. □

Thus, it makes again good sense to speak of the quasivariety $\mathrm{qvar}(F)$ generated by a non-cyclic free matabelian group.

3. $\mathcal{A}$-rings

Definition. A ring S is called an $\mathcal{A}$-ring if

(i) S is a factor ring of R_n for some natural number n,

(ii) $\mathrm{Ass}(S) = \{\Delta_{B_1}, \Delta_{B_2}, \ldots, \Delta_{B_k}\}$ with $B_1, B_2, .., B_k \in \mathcal{P}(A_n)$,

(iii) S is semisimple.

Lemma 3.1. *Every primary* $\mathcal{A}$*-ring is isomorphic to* R_m *for some natural number* m.

Proof. Let $S = R_n/I$ for some natural number n and some ideal $I \lhd R_n$, and let $\mathrm{Ass}(S) = \{\Delta_B\}$ with $B \in \mathcal{P}(A_n)$. Then $\Delta_B^l \subseteq I \subseteq \Delta_B$ for some natural number l. Since S has no nilpotent elements, we must have $I = \Delta_B$. But then $S \cong R_m$ where $m = n - \mathrm{rank}(B)$. $\square$

Lemma 3.2. *Let S be an arbitrary $\mathcal{A}$-ring. Then there exist natural numbers $n_1, n_2, \ldots, n_q$ such that S is the subdirect sum of the rings $R_{n_1}, R_{n_2}, \ldots, R_{n_q}$.*

Proof. Let $S = R_n/I$ with $\mathrm{Ass}(S) = \{\Delta_{B_1}, \ldots, \Delta_{B_k}\}$ and let $I = I_1 \cap I_2 \cap \ldots \cap I_q$ be a primary decomposition of the ideal I. Then, for each j $(1 \leq j \leq q)$ there are natural numbers $i_j \in \{1, \ldots, k\}$ and $l_j \geq 1$ such that $I_j \subseteq \Delta_{B_{i_j}} \in \mathrm{Ass}(S)$ and $\Delta_{B_{i_j}}^{l_j} \subseteq I \subseteq \Delta_{B_{i_j}}$. We claim that, in fact,

$$I = \Delta_{B_{i_1}} \cap \Delta_{B_{i_2}} \cap \ldots \cap \Delta_{B_{i_q}}. \tag{3.1}$$

Indeed, if $\alpha \in \Delta_{B_{i_1}} \cap \Delta_{B_{i_2}} \cap \ldots \cap \Delta_{B_{i_q}}$, then $\alpha^l \in I$ for $l = \Pi_{j=1}^q l_{i_j}$, and since S has no nilpotent elements this gives $\alpha \in I$. But (3.1) guarantees that S is isomorphic to the subdirect sum of the rings $R/\Delta_{B_{i_j}}$ $(j = 1, \ldots, q)$, and since $R/\Delta_{B_{i_j}} \cong R_{n_j}$ with $n_j = n - \mathrm{rank}(B_{i_j})$, the Lemma follows. $\square$

Definition. An ideal I of an $\mathcal{A}$-ring S is termed a radical ideal if the quotient ring S/I is again an $\mathcal{A}$-ring.

Our next aim is to show that the intersection of any family of radical ideals in an $\mathcal{A}$-ring is again a radical ideal, the main result of this section. We need some auxiliary results.

Lemma 3.3. *Let $B_1, B_2, \ldots, B_s \in \mathcal{P}(A_n)$. Then the intersection $I = \bigcap_{i=1}^s \Delta_{B_i}$ is a radical ideal in R_n.*

Proof. Let $S = R_n/I$. Then $\mathrm{Ass}(S) = \{\Delta_{B_1}, \ldots, \Delta_{B_s}\}$ and the ring S is semisimple. Hence S is an $\mathcal{A}$-ring. $\square$

Recall the definition of $C(\alpha)$, the contents of $\alpha \in R_n$ (see Section 2.2).

Lemma 3.4. *Let $B \in \mathcal{P}(A_n)$ and $\alpha \in \Delta_B$ with $\alpha \neq 0$. Then there exists an element $b \in C(\alpha)$ such that $b \in B$.*

Proof. Consider the factor ring R_n/Δ_B, and, for any $\beta \in R_n$, let $\overline{\beta}$ denote the image of β under the natural homomorphism $R_n \to R_n/\Delta_B$.

Since $\alpha \in \Delta_B$, we have $\overline{\alpha} = 0$. Then, for any $g \in \operatorname{supp}^+(\alpha)$, there exists a $h \in \operatorname{supp}^-(\alpha)$ such that $\overline{g} = \overline{h}$. Consequently, $gh^{-1} \in B$, and since B is a pure subgroup, the element $b \in C(\alpha)$ with $b^l = gh^{-1}$ also belongs to B. $\square$

Now we introduce the associator $\operatorname{Ass}(\alpha)$ of an element $\alpha \in R_n$ as follows. If $\alpha \notin \Delta_{A_n}$, we set $\operatorname{Ass}(\alpha) = \emptyset$, and if $\alpha \in \Delta_{A_n}$, we define $\operatorname{Ass}(\alpha)$ as the set of all $B \in \mathcal{P}(A_n)$ such that $\alpha \in \Delta_B$, but $\alpha \notin \Delta_C$ for all proper pure subgroups C of B. Thus

$$\operatorname{Ass}(\alpha) = \{B \in \mathcal{P}(A_n)\ ;\ \alpha \in \Delta_B,\ \alpha \notin \Delta_C\ \forall\, C \in \mathcal{P}(B) \setminus B\}.$$

Lemma 3.5. *For all $\alpha \in \Delta_{A_n}$, $\operatorname{Ass}(\alpha)$ is a finite set.*

Proof. We use induction on n. The lemma is trivial if $n = 1$, so let $n > 1$, and suppose that for some $\alpha \in \Delta_{A_n}$ the associator $\operatorname{Ass}(\alpha) = \{B_i\ ,\ i \in I\}$ consists of infinitely many distinct subgroups $B_i \in \mathcal{P}(A_n)$. By Lemma 3.4, each B_i contains an element of the finite set $C(\alpha)$, and hence there is a $b \in C(\alpha)$ that is contained in infinitely many of the B_i, say $b \in B_i$ for all $i \in J$ where J is an infinite subset of I. Consider the natural homomorphism $\varphi : A_n \to A_n/\langle b\rangle$. Since b is a primitive element, $A_n/\langle b\rangle$ is a free abelian group of rank $n-1$. It is clear that the images $B_i\varphi$ $(i \in J)$ are distinct in $A_n/\langle b\rangle$. Now let $\varphi : \mathbb{Z}A_n \to \mathbb{Z}(A_n/\langle b\rangle)$ denote the corresponding homomorphism of integral group rings, and let $\overline{\alpha} = \alpha\varphi$. Clearly, $\overline{\alpha} \in \Delta_{A_n/\langle b\rangle}$. Moreover, it is plain that, if $B \in \operatorname{Ass}(\alpha)$, then $B\varphi \in \operatorname{Ass}(\overline{\alpha})$, and hence the latter contains the infinite set $\{B_i\varphi\ ,\ i \in J\}$. Since $\mathbb{Z}(A_n/\langle b\rangle) \cong \mathbb{Z}A_{n-1}$, this contradicts the inductive hypothesis. The lemma follows. $\square$

Definition. (The radical of a non-zero element $\alpha \in R_n$) Let $\alpha \in R_n$ $(\alpha \neq 0)$. If $\alpha \notin \Delta_{R_n}$ we set $\operatorname{Rad}(\alpha) = R_n$, and if $\alpha \in \Delta_{R_n}$ we set

$$\operatorname{Rad}(\alpha) = \bigcap_{B \in \operatorname{Ass}(\alpha)} \Delta_B.$$

Note that, by Lemma 3.3, $\operatorname{Rad}(\alpha)$ is a radical ideal.

Lemma 3.6. *The radical $\operatorname{Rad}(\alpha)$ is the smallest radical ideal of R_n containing α.*

Proof. Let I be a radical ideal of R_n, and let $\alpha \in I$. Then $S = R_n/I$ is an $\mathcal{A}$-ring. In the proof of Lemma 3.2 we have seen that either $I = R_n$ or $I = \Delta_{B_{i_1}} \cap \ldots \cap \Delta_{B_{i_q}}$ with $\Delta_{B_{i_1}}, \ldots, \Delta_{B_{i_q}} \in \operatorname{Ass}(S)$. But if $\alpha \in \Delta_{B_{i_j}}$

then B_{i_j} contains a subgroup $C \in \operatorname{Ass}(\alpha)$, and hence $\Delta_{B_{i_j}} \supseteq \Delta_C$. Consequently,

$$I = \Delta_{B_{i_1}} \cap \ldots \cap \Delta_{B_{i_q}} \supseteq \bigcap_{C \in \operatorname{Ass}(\alpha)} \Delta_C = \operatorname{Rad}(\alpha),$$

and the lemma follows. □

The following technical result will be used in the proof of Proposition 3.1 below.

Lemma 3.7. *Let $\alpha = n_1 g_1 + \cdots + n_l g_l \in R_n$ with $\operatorname{supp}(\alpha) = \{g_1, \ldots, g_l\}$ and $l > 1$, and let k be a positive integer with $k > |n_1| + \cdots + |n_l|$. Then the equation $x^k = \alpha$ has no solution in R_n.*

Proof. Since the infinite cyclic group $A_1 = \langle a \rangle$ discriminates A_n, there is a homomorphism $\varphi : R_n \to R_1 = \mathbb{Z}A_1$ such that

$$\alpha\varphi = n_1 a^{\lambda_1} + \cdots + n_l a^{\lambda_l}$$

where $\lambda_1, \ldots, \lambda_l$ are pairwise distinct integers. Without loss of generality we may assume that $\lambda_1 < \cdots < \lambda_l$. Suppose that there is a $\beta \in R_n$ such that $\beta^k = \alpha$. Then $(\beta\varphi)^k = \alpha\varphi$. Let

$$\beta\varphi = m_1 a^{\nu_1} + \cdots + m_q a^{\nu_q}$$

where $m_1, \ldots, m_q$ are non-zero integer coefficients and $\nu_1 < \cdots < \nu_q$. Then $(\beta\varphi)^k = \alpha\varphi$ implies that $\lambda_1 = k\nu_1$ and $\lambda_2 = (k-1)\nu_1 + \nu_2$. The coefficients of $\alpha\varphi$ and $(\beta\varphi)^k$ at $a^{\lambda_2} = a^{(k-1)\nu_1 + \nu_2}$ are n_2 and $km_1^{k-1}m_2$, respectively. But our assumption on k gives $n_2 \neq km_1^{k-1}m_2$, so $(\beta\varphi)^k \neq \alpha\varphi$, a contradiction. The lemma follows. □

Proposition 3.1. *Let Y be a finite set of elements in R_n. Then there exists a minimal radical ideal of R_n containing Y. This ideal will be called the radical of Y, and denoted by* $\operatorname{Rad}(Y)$*.*

Proof. We need to establish the existence of a minimal radical ideal containing Y. It is sufficient to consider the case where $Y = \{\alpha, \beta\}$ consists of two non-zero elements with $\alpha, \beta \in \Delta_{A_n}$. Then $|\alpha| \geq 2$. Let $\alpha = n_1 g_1 + \cdots + n_l g_l \in R_n$ with $\operatorname{supp} \alpha = \{g_1, \ldots, g_l\}$ $(l > 1)$, let k be a positive integer with $k > |n_1| + \cdots + |n_l|$, and consider the element $\gamma = \alpha - \beta^k$. If I is a radical ideal with $\alpha, \beta \in I$, then $\gamma \in I$, and hence $\operatorname{Rad}(\gamma) \subseteq I$ by Lemma 3.6. It is therefore sufficient to show that $\alpha, \beta \in \operatorname{Rad}(\gamma)$. This is clear when $\operatorname{Rad}(\gamma) = R_n$. Now suppose that $\gamma \in \Delta_{A_n}$, and let $B \in \operatorname{Ass}(\gamma)$. Let $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ denote the images of α, β, γ

under the natural homomorphism $R_n \to R_n/\Delta_B$. Then $\overline{\alpha} - \overline{\beta}^k = \overline{\gamma} = 0$ (since $\gamma \in \Delta_B$), and hence $\overline{\alpha} = \overline{\beta}^k$ in R_n/Δ_B. The latter is isomorphic to R_m where $m = n - \operatorname{rank} B$. It is easily seen that, if $\overline{\alpha} \neq 0$, then $\overline{\alpha}$ and k satisfy the conditions of Lemma 3.7 (applied in R_m), and the equation $\overline{\alpha} = \overline{\beta}^k$ contradicts the conclusion of this lemma. Consequently, we must have $\overline{\alpha} = 0$. But then $\overline{\beta} = 0$ as well, and hence $\alpha, \beta \in \Delta_B$ for all $B \in \operatorname{Ass}(\gamma)$. This gives that $\alpha, \beta \in \operatorname{Rad}(\gamma)$ as required. □

Corollary 3.1. *Let Y be an arbitrary subset of R_n. Then there exists a minimal radical ideal $\operatorname{Rad}(Y)$ of R_n that contains Y. Moreover, there exists an element $\gamma \in R_n$ such that $\operatorname{Rad}(Y) = \operatorname{Rad}(\gamma)$.*

Proof. The case where Y is a finite set is covered by the Proposition 3.1. Now suppose that Y is infinite. Since Y is countable, it can be ordered (of type ω): $Y = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$. Let $Y_i = \{\alpha_1, \ldots, \alpha_i, \}$, and let $\operatorname{Rad}(Y_i) = \operatorname{Rad}(\gamma_i)$, $i = 1, 2, \ldots$. Then $\operatorname{Rad}(Y_1) \subseteq \operatorname{Rad}(Y_2) \subseteq \ldots$, and the corollary follows since R_n is Noetherian. □

Now we are ready for the main result of this section.

Theorem 3.1. *Let S be an $\mathcal{A}$-ring, and let I_i $(i \in J)$ be a family of radical ideals in S. Then the intersection $I = \bigcap_{i \in J} I_i$ is also a radical ideal in S.*

Proof. Suppose that the $\mathcal{A}$-ring S is a factor ring of R_n. It is easily seen that an ideal I of S is radical if and only if its full inverse image under the surjection $R_n \to S$ is a radical ideal in R_n. It is therefore sufficient to prove the theorem in the case where $S = R_n$. Moreover, by Lemma 3.2, it is sufficient to consider the case where $I_i = \Delta_{B_i}$ $(i \in J)$ with $B_i \in \mathcal{P}(A_n)$. Let $C = \bigcap_{i \in J} B_i$. Then C is a pure subgroup of A_n, that is $C \in \mathcal{P}(A_n)$. If $C \neq 1$, then $\Delta_C \subseteq \Delta_{B_i}$ for all $i \in J$, and hence it is sufficient to prove the Theorem in the factor ring $R_n/\Delta_C \cong R_{n-\operatorname{rank} C}$. We may assume that $C = 1$.

Now, if $I = \{0\}$, the theorem is true since $\{0\}$ is a radical ideal. If $I \neq \{0\}$, the theorem will be proved once we show that $I = \operatorname{Rad}(I)$. By Proposition 3.1 and the corollary thereafter, we have that $\operatorname{Rad}(I) = \operatorname{Rad}(\gamma)$ for some $\gamma \in I$. Hence, for any $i \in J$, $\gamma \in I_i$, and $\operatorname{Rad}(\gamma) \subseteq I_i$ (since I_i is a radical ideal). Therefore $\operatorname{Rad}(\gamma) \subseteq \bigcap_{i \in J} I_i$, and the theorem follows. □

We conclude this section with the following

Proposition 3.2. *For any finite set Y in R_n, there is an algorithm for computing a generating set for* $\mathrm{Rad}(Y)$ *as an ideal.*

Proof. By Proposition 3.1, $\mathrm{Rad}(Y) = \mathrm{Rad}(\gamma)$ for some $\gamma \in R_n$. Moreover, the proof of Proposition 3.1 provides an algorithm for determining such a γ for any given Y. Once we have γ, we can determine the contents $C(\gamma)$, and then we can compute the associator $\mathrm{Ass}(\gamma) = \{B_1, \dots, B_k\}$. More precisely, we can find free generating sets for the free abelian groups $B_1, \dots, B_k$ (and hence generating sets for the ideals $\Delta_{B_1}, \dots, \Delta_{B_k}$). Indeed, by Lemma 3.4, any $B \in \mathrm{Ass}(\gamma)$ contains an element b of the finite set $C(\gamma)$, and then we can work in $\mathbb{Z}(A_n/\langle b\rangle$, the group ring of a free abelian group of smaller rank (details are left to the reader). By definition, $\mathrm{Rad}(\gamma) = \bigcap_{i=1}^{k} \Delta_{B_i}$, and now it remains to appeal to a result by Seidenberg [10] which says that there is an algorithm for determining the generators of the intersection of a finite set of ideals in R_n from the generators of those ideals. □

4. $\mathcal{A}$-modules

Definition. A finitely generated R_n-module M is called an $\mathcal{A}$-module, if it satisfies the following two conditions

(i) for every non-zero element $u \in M$, the factor ring $R_n/\operatorname{Ann}(u)$ is an $\mathcal{A}$ -ring,

(ii) for any subgroup $C \in \mathcal{P}(A_n)$, $M\Delta_C \cap M[\Delta_C] = \{0\}$.

(See Section 2.2 for the definition of $M[\Delta_C]$.)

Remark 4.1. In view of the results of Section 3, the first condition in the definition of an $\mathcal{A}$-module is equivalent to either of the following.

(i') For any $u \in M$, either $\operatorname{Ann}(u) = R_n$ or $\operatorname{Ann}(u) = \bigcap \Delta_{B_i}$ for suitable $B_1, \dots, B_k \in \mathcal{P}(A_n)$.

(i") For any $u \in M$, the annihilator $\operatorname{Ann}(u)$ is a radical ideal, and hence there is an element $\gamma \in R_n$ such that $\operatorname{Ann}(u) = \mathrm{Rad}(\gamma)$.

Lemma 4.1. *For any primary $\mathcal{A}$ -module M there exists a unique natural number n such that M is a torsion-free R_n-module, and, conversely, any finitely generated torsion-free R_n-module M is an $\mathcal{A}$ -module.*

Proof. Suppose M is a primary R_m-module with $\mathrm{Ass}(M) = \{\Delta_B\}$ with $B \in \mathcal{P}(A_m)$. Then for every non-zero element $u \in M$, the factor ring $R_k/\operatorname{Ann}(u)$ is a primary $\mathcal{A}$-ring. By Lemma 3.1, $S \cong R_n$ where

$n = m - \mathrm{rank}(B)$, and hence M is a torsion-free R_n-module as required. The second part of the lemma is trivial. □

The main result of this section is the following

Theorem 4.1. *Let M be an arbitrary $\mathcal{A}$-module. Then there exist natural numbers $n_1, n_2, \ldots, n_k$ such that M is the subdirect sum of modules $M_1, M_2, \ldots, M_k$ where each M_i $(i = 1, 2, \ldots, k)$ is a torsion-free module for R_{n_i}.*

Before we embark on the proof of this theorem, we recall some facts about the primary decomposition of the module M, and prove some auxiliary results. Let M be an $\mathcal{A}$-module over R_n with

$$\mathrm{Ass}(M) = \{\Delta_{B_1}, \ldots, \Delta_{B_k}\} \tag{4.1}$$

where $B_1, \ldots, B_k \in \mathcal{P}(A_n)$. Then there is a reduced primary decomposition of the zero-submodule of M of the form

$$\{0\} = \bigcap_{i=1}^{k} N_i \quad \text{with} \quad \mathrm{Ass}(M/N_i) = \Delta_{B_i}. \tag{4.2}$$

Moreover, for each i $(1 \le i \le k)$, one has $\mathrm{Ass}(N_i) = \bigcup_{j \neq i}\{\Delta_{B_j}\}$ and $\bigcap_{j \neq i} N_j \neq \{0\}$ (see [2, Chapter 4, §2, Proposition 4]).

Definition. Let M be an $\mathcal{A}$-module. We say that a primary decomposition $N = \bigcap_{i=1}^{l} Q_i$ of a submodule N of M is an $\mathcal{A}$-primary decomposition if all primary factor modules M/Q_i $(i = 1, \ldots, l)$ are themselves $\mathcal{A}$-modules.

In view of Lemma 4.1, Theorem 4.1 will be proved once we show that the zero-submodule of M has an $\mathcal{A}$-primary decomposition $\{0\} = \bigcap_{i=1}^{l} Q_i$, i.e. for all non-zero elements $u \in M/Q_i$, one has $\mathrm{Ann}(u) = \Delta_{B_i}$. Indeed, in this case M/Q_i is a torsion-free module over $\overline{R} = R_n/\Delta_{B_i} \cong R_{n_i}$ where $n_i = n - \mathrm{rank}(B_i)$.

Now let $M_i = M/N_i$ $(i = 1, \ldots, k)$, where the N_i are as in (4.2). We will assume that the module M is canonically embedded in the direct sum $M_1 \oplus \cdots \oplus M_k$, and set $M^{(i)} = M \cap M_i$.

Lemma 4.2. *Suppose that Δ_{B_1} is a maximal (with respect to inclusion) ideal in $\mathrm{Ass}(M)$. Then $M[\Delta_{B_1}] = M^{(1)}$.*

Proof. Let $0 \neq u \in M[\Delta_{B_1}]$. Since Δ_{B_1} is a maximal ideal, there exists an element $\alpha \in \Delta_{B_1}$ that is not contained in the other ideals in

$\operatorname{Ass}(M)$. Let $u = (u_1, \dots, u_k)$ and suppose that $u_i \neq 0$. If $i \neq 1$, then $u_i\alpha \neq 0$ in M_i since $\alpha \notin \Delta_{B_i}$, and, consequently, multiplication by α is an injective map on M_i. It follows that $u_2 = \dots = u_k = 0$ and so $u = (u_1, 0, \dots, 0) \in M^{(1)}$.

Now let $u = (u_1, 0, \dots, 0) \in M^{(1)}$. Since $\alpha \in \Delta_{B_1}$, multiplication by α is a nilpotent map on M_1. Hence $u\alpha^t = 0$ for some natural number t, so $\alpha^t \in \operatorname{Ann}(u)$. But since $\operatorname{Ann}(u)$ is the intersection of prime ideals, it follows that $\alpha \in \operatorname{Ann}(u)$, and, consequently, $u \in M[\Delta_{B_1}]$. □

Now consider the factor module $M/M^{(1)}$. Since $M^{(1)} = \bigcap_{i=2}^{k} N_i$, this is a reduced primary decomposition of $M^{(1)}$ and $M_2, \dots, M_k$ are the corresponding primary components for $M/M^{(1)}$. If we can show that $M/M^{(1)}$ is an $\mathcal{A}$-module, Theorem 4.1 can be deduced relatively easily from this fact.

Let $\overline{u}$ be a non-zero element in $M/M^{(1)}$. Take $u \in M$ such that $\overline{u} = u + M^{(1)}$. Let $\operatorname{Ann}(u) = \bigcap_{j \in J} \Delta_{B_j}$ where J is a subset of $\{1, 2, \dots, k\}$.

Lemma 4.3. *In the notation introduced above,* $\operatorname{Ann}(\overline{u}) = \bigcap_{j \in J_0} \Delta_{B_j}$, *where* $J_0 \subseteq J$, *and if* $j \in J_0$, *then* Δ_{B_j} *does not contain* Δ_{B_1}.

Proof. First we show that

$$\operatorname{Ann}(\overline{u}) = (\operatorname{Ann}(u) : \Delta_{B_1})$$

(see Section 2.2 for the definition of $(\operatorname{Ann}(u) : \Delta_{B_1})$). Indeed, suppose that $\beta \in \operatorname{Ann}(\overline{u})$. Then $u\beta = u' \in M^{(1)}$. But $M^{(1)} = M[\Delta_{B_1}]$ by Lemma 4.2. Hence $u'\gamma = u\beta\gamma = 0$ for all $\gamma \in \Delta_{B_1}$, in other words $\beta \in (\operatorname{Ann}(u) : \Delta_{B_1})$. Conversely, if $\beta \in (\operatorname{Ann}(u) : \Delta_{B_1})$, then $u\beta\gamma = 0$ for all $\gamma \in \Delta_{B_1}$, which is the same as to say that $u\beta \in M[\Delta_{B_1}]$. Again, by Lemma 4.2, this gives $u\beta \in M^{(1)}$, and hence $\beta \in \operatorname{Ann}(\overline{u})$.

Our next aim is to determine the ideal $(\operatorname{Ann}(u) : \Delta_{B_1})$. Since $\operatorname{Ann}(u) \subseteq \operatorname{Ann}(\overline{u})$, it is sufficient to calculate $(\{0\} : \overline{\Delta}_{B_1})$ in the factor ring $\overline{R} = R_n / \operatorname{Ann}(u)$, and then to take the full inverse image in R_n. For, we consider the canonical embedding of $\overline{R}$ into $\bigoplus_{j \in J} R_n/\Delta_{B_j}$. If $j \in J_0$, the image of Δ_{B_1} is a non-zero ideal in R_n/Δ_{B_j}, and hence, if $\alpha \in (\{0\} : \overline{\Delta}_{B_1})$, then $\alpha \in \Delta_{B_j}$, and, consequently, $(\{0\} : \overline{\Delta}_{B_1}) \subseteq \bigcap_{j \in J_0} \Delta_{B_j}$. On the other hand, if $\alpha \in \bigcap_{j \in J_0} \Delta_{B_j}$, then $\alpha \in \overline{\Delta}_{B_1}$ in $\overline{R}$. □

Lemma 4.4. *The module* $\overline{M} = M/M^{(1)}$ *is an* $\mathcal{A}$*-module.*

Proof. By Lemma 4.3, $\overline{M}$ satisfies the condition (i') in Remark 4.1, which is equivalent to condition (i) in the definition of an $\mathcal{A}$-module. It is therefore sufficient to check that $\overline{M}$ satisfies condition (ii) in that definition. Let C be a non-trivial pure subgroup of A_n, let $\overline{u} \in \overline{M}\Delta_C \cap \overline{M}[\Delta_C]$, and let u be an inverse image of $\overline{u}$ in M. Since $\overline{u} \in \overline{M}\Delta_C$, we have $u = u_1 + u_2$ where $u_1 \in M\Delta_C$ and $u_2 \in M^{(1)} = M[\Delta_{B_1}]$. Consequently, for all $b \in B_1$, we have $u(b-1) = u_1(b-1) \in M\Delta_C$. On the other hand, since $\overline{u} \in \overline{M}[\Delta_C]$, we have $u(c-1) \in M^{(1)} = M[\Delta_{B_1}]$ for all $c \in C$. But then $u(c-1)(b-1) = 0$ for all $b \in B_1$. Hence $u(b-1) \in M[\Delta_C]$. It follows that $u(b-1) \in M\Delta_C \cap M[\Delta_C]$ for all $b \in B_1$. But since M is an $\mathcal{A}$-module, this intersection is zero. Hence $u \in M[\Delta_{B_1}] = M^{(1)}$ and therefore $\overline{u} = 0$. □

Now we turn to the proof of Theorem 4.1.

Proof. Let M be an $\mathcal{A}$-module over R_n with $\mathrm{Ass}(M)$ as in (4.1) and with a reduced primary decomposition of the zero-submodule as in (4.2). We show by induction on $k = |\mathrm{Ass}(M)|$ that the zero-submodule of M admits an $\mathcal{A}$-primary decomposition $\{0\} = \bigcap_{i=1}^{k} Q_i$. If $k = 1$, this is ensured by Lemma 4.1. Now let $k > 1$. We may assume that Δ_{B_1} is a maximal (with respect to inclusion) element in the partially ordered set $\mathrm{Ass}(M)$. Then, by Lemma 4.3 and Lemma 4.4, the factor module $\overline{M} = M/M^{(1)}$ is an $\mathcal{A}$-module, $\mathrm{Ass}(\overline{M}) \subset \mathrm{Ass}(M)$, and $|\mathrm{Ass}(\overline{M})| = k-1$. Hence, by induction, for the submodule $M^{(1)}$ there exists the required primary decomposition $\{M^{(1)}\} = \bigcap_{i=2}^{k} Q_i$. If there is another maximal element in the partially ordered set $\mathrm{Ass}(M)$, Δ_{B_2} say, we consider the reduced primary decomposition $\{0\} = N_1 \cap \bigcap_{i=2}^{k} Q_i$, the submodule $M^{(2)}$ and the factor module $\widetilde{M} = M/M^{(2)}$ (with regard to this new primary decomposition). By Lemma 4.4, $\widetilde{M}$ is an $\mathcal{A}$-module, and hence there is an $\mathcal{A}$-primary decomposition $M^{(2)} = Q_1' \cap Q_3' \cap \cdots \cap Q_k'$. But then $\{0\} = Q_1' \cap Q_2 \cap Q_3' \cap \cdots \cap Q_k'$ is an $\mathcal{A}$-primary decomposition for M.

Now suppose that Δ_{B_1} is the only maximal element in $\mathrm{Ass}(M)$, so $\Delta_{B_i} \subseteq \Delta_{B_1}$ for $i = 1, \ldots, k$. Using the $\mathcal{A}$-primary decomposition $M^{(1)} = \bigcap_{i=2}^{k} Q_i$, we construct an $\mathcal{A}$-primary decomposition for M. For, we set $D = R_n \setminus \Delta_{B_1}$ and we denote by Q_1' the submodule generated by the elements $\frac{v}{\alpha}$ with $v \in M\Delta_{B_1}$ and $\alpha \in D$, where $\frac{v}{\alpha}$ denotes the unique solution of the equation $x\alpha = v$ if it exist in M. This is the so-called D-isolator of the submodule $M\Delta_{B_1}$ in M. Let $M_1 = M/Q_1'$.

We now show that M_1 is a primary $\mathcal{A}$-module. Let $\overline{u} = u + Q'_1$ be a non-zero element of M_1. First of all we show that $\mathrm{Ann}(\overline{u}) = \Delta_{B_1}$. It is clear that $\mathrm{Ann}(\overline{u}) \supseteq \Delta_{B_1}$ since $Q'_1 \supseteq M\Delta_{B_1}$. If there exists an element $\alpha \in \mathrm{Ann}(\overline{u}) \backslash \Delta_{B_1}$, then $\overline{u}\alpha = 0$, and hence $u\alpha \in Q'_1$. Let $u\alpha = v$. Then $u = \frac{v}{\alpha}$ and $u \in Q'_1$ by the definition of that submodule. It follows that M_1 is a torsion free module over the ring R_n/Δ_{B_1}, and now Lemma 4.1 gives that M_1 is a primary $\mathcal{A}$-module.

Finally, we check that $Q'_1 \cap M^{(1)} = \{0\}$. Suppose $u \in Q'_1 \cap M^{(1)}$. For any element u in Q'_1 there exists an element $\alpha \in D$ such that $u\alpha \in M\Delta_{B_1}$. It is clear that $u\alpha \in M^{(1)} = M[\Delta_{B_1}]$. Since $M[\Delta_{B_1}] \cap M\Delta_{B_1} = \{0\}$, we have $u\alpha = 0$. Now, if $u \neq 0$, then $\alpha \in \mathrm{Ann}(u)$ and $\alpha \in \Delta_{B_1}$, which is not the case. It follows that $u = 0$, and $\{0\} = Q'_1 \cap Q_2 \cap Q'_3 \cap \cdots \cap Q'_k$ is an $\mathcal{A}$-primary decomposition. This completes the proof of the theorem. □

Proposition 4.1. *Let M_i $(i = 1, \ldots, q)$ be $\mathcal{A}$-modules over $R_n = \mathbb{Z}A_n$. Then*

(i) *each M_i is an $\mathcal{A}$-module for any $R_m = \mathbb{Z}A_m$ with $m > n$ and $A_m = A_n \oplus C$, provided that C acts trivially on M_i,*

(ii) *$M = M_1 \oplus \cdots \oplus M_q$ is an $\mathcal{A}$-module over R_n,*

(iii) *any submodule N of $M = M_1 \oplus \cdots \oplus M_q$ is an $\mathcal{A}$-module.*

Proof. (i) Let u be a non-zero element of M_i. Then it is easily seen that if $\mathrm{Ann}_{R_n}(u) = \bigcap_{j=1}^{l} \Delta_{B_j}$ with $B_j \in \mathcal{P}(A_n)$, then $\mathrm{Ann}_{R_m}(u) = \bigcap_{j=1}^{l} \Delta_{B_j \oplus C}$. Now we check the second condition in the definition of an $\mathcal{A}$-module. Let D be a subgroup of A_m, and let D_0 denote the projection of D onto A_n. If $D_0 = \{0\}$, then $D \leq C$, and in this case we have $M_i\Delta_D = \{0\}$. Hence M_i is an $\mathcal{A}$-module. If $D_0 \neq \{0\}$, then $M_i\Delta_D = M_i\Delta_{D_0}$ and $M_i[\Delta_D] = M_i[\Delta_{D_0}]$. Hence $M_i\Delta_D \cap M_i[\Delta_D] = M_i\Delta_{D_0} \cap M_i[\Delta_{D_0}] = \{0\}$, and we have again that M_i is an $\mathcal{A}$-module.

(ii) Let $u = (u_1, \ldots, u_q)$ be non-zero element of M. Then $\mathrm{Ann}(u) = \bigcap_{i=1}^{q} Ann(M_i)$, and hence the first condition for $\mathcal{A}$-modules holds. Now let D be a subgroup of A_n. Then $M\Delta_D = M_1\Delta_D \oplus \ldots \oplus M_q\Delta_D$ and $M[\Delta_D] = M_1[\Delta_D] \oplus \ldots \oplus M_q[\Delta_D]$. Since for any i $(1 \leq i \leq q)$, $M_i\Delta_D \cap M_i[\Delta_D] = \{0\}$, the second condition holds as well.

(iii) Let $0 \neq u \in N$. Then the first condition holds since it depends only on u, and not on N. It is clear that the second condition holds as well. □

Proposition 4.2. *Let M be an $\mathcal{A}$-module over R_n, and C a subgroup of A_n. Then any finitely generated $\mathbb{Z}C$-submodule N of M is an $\mathcal{A}$-module over $\mathbb{Z}C$.*

Proof. Let u be a non-zero element of N. Then $\mathrm{Ann}_{\mathbb{Z}C}(u) = \mathrm{Ann}_{R_n}(u) \cap \mathbb{Z}C$. Let $\mathrm{Ann}_{R_n}(u) = \bigcap_{i=1}^{l} \Delta_{B_i}$ with $B_i \in \mathcal{P}(A_n)$. Then $\Delta_{B_i} \cap \mathbb{Z}C = \Delta_{B_i \cap C}$. Indeed,

$$\mathbb{Z}C/(\Delta_{B_i} \cap \mathbb{Z}C) \cong (\mathbb{Z}C + \Delta_{B_i})/\Delta_{B_i} \cong \mathbb{Z}C/\Delta_{B_i \cap C}.$$

Since

$$(\bigcap_{i=1}^{l} \Delta_{B_i}) \cap \mathbb{Z}C = \bigcap_{i=1}^{l} (\Delta_{B_i} \cap \mathbb{Z}C) = \bigcap_{i=1}^{l} \Delta_{B_i \cap C}$$

and since the subgroup $B_i \cap C$ is pure in C (because $C/(B_i \cap C)$ is torsion-free), the first condition in the definition of an $\mathcal{A}$-module holds. If D is a subgroup of C, then $N\Delta_D \cap N[\Delta_D] \subseteq M\Delta_D \cap M[\Delta_D]$, and hence the second condition holds as well. □

5. Metabelian $\mathcal{A}$-groups

Definition. A group G is called an $\mathcal{A}$ -group if the following four conditions hold.

(i) G is a finitely generated torsion-free metabelian group,

(ii) G has no non-abelian nilpotent subgroups, and hence the Fitting subgroup $\mathrm{Fit}(G)$ is abelian,

(iii) the quotient $G/\mathrm{Fit}(G)$ is a free abelian group:

$$G/\mathrm{Fit}(G) \cong A_n$$

for some n, and as a module for $\mathbb{Z}A_n$, $\mathrm{Fit}(G)$ is an $\mathcal{A}$-module,

(iv) For any finitely generated subgroup H of G, $Z(H) \cap H' = 1$.

It will be convenient to use the following terminology which mimics our terminology for modules. For an $\mathcal{A}$-group G we define $\mathrm{Ass}(G)$, the associator of G, to be the associator of $\mathrm{Fit}(G)$ as an $\mathbb{Z}A_n$-module. We say that an $\mathcal{A}$-group G is a primary $\mathcal{A}$-group if $\mathrm{Ass}(G)$ consists of a single ideal.

Proposition 5.1. *Every primary $\mathcal{A}$-group is a ρ-group and, conversely, every finitely generated ρ-group is an $\mathcal{A}$-group.*

Proof. Let G be a primary $\mathcal{A}$-group with $\mathrm{Ass}(G) = \{\Delta_B\}$ where B is a pure subgroup of A_n. If $B \neq 1$, then for any $b \in B$ with $b \neq 1$

and any $u \in \mathrm{Fit}(G)$ one has $u^{(1-b)} = 1$. Let g be an inverse image of b in G. Then $\langle \mathrm{Fit}(G), g\rangle$ is a normal subgroup in G, and hence $g \in \mathrm{Fit}(G)$, contradicting our assumption that $b \neq 1$. Hence $\mathrm{Ass}(G) = \{0\}$. Consequently, $\mathrm{Fit}(G)$ is a torsion-free $\mathbb{Z}A_n$-module, and hence G is a ρ-group. For the second part of the proposition, let G be a finitely generated ρ-group and $M = \mathrm{Fit}(G)$. Then $G/\mathrm{Fit}(G) \cong A_n$ for some n, and since M is a torsion-free $\mathbb{Z}A_n$-module, it is an $\mathcal{A}$-module by Lemma 4.1. Let H be a finitely generated subgroup of G. If H is abelian, $H' = 1$, and hence $Z(H) \cap H' = 1$. If H is not abelian, then $Z(H) = 1$ since the axiom CT holds in H (see [3, Lemma 3.7]). □

Proposition 5.2. *A direct product of finitely many $\mathcal{A}$-groups is an $\mathcal{A}$-group.*

Proof. Let $G = G_1 \times \cdots \times G_k$ where $G_1, \ldots, G_k$ are $\mathcal{A}$-groups. Let $M_i = \mathrm{Fit}(G_i)$ and $A_i = G_i/M_i$. Then $M = M_1 \times \ldots \times M_k$ and $G/M \cong A_1 \times \ldots \times A_k$. It is easy to see that conditions (i),(ii),(iv), and the first part of condition (iii) in the definition of an $\mathcal{A}$-group hold for G, and the second part of condition (iii) follows from Proposition 4.6. □

Examples. The following are examples of $\mathcal{A}$-groups:
(i) the free abelian groups A_n,
(ii) the wreath products $W_{r,s} = A_r \mathrm{wr} A_s$,
(iii) the free metabelian group F,
(iv) direct products of the form $W = W_{r_1,s_1} \times \ldots \times W_{r_k,s_k}$.

The wreath products (ii) are $\mathcal{A}$-groups by Proposition 5.1 because they are ρ-groups (as we have noticed in Section 2.1). The fact that F is an $\mathcal{A}$-group follows (in view of the Magnus embedding) immediately from Proposition 5.3 below, but it can also be easily verified directly. The groups (iv) are $\mathcal{A}$-groups by Proposition 5.2. They will be referred to as *canonical* $\mathcal{A}$-groups.

Proposition 5.3. *Any finitely generated subgroup of a canonical $\mathcal{A}$-group W is itself an $\mathcal{A}$-group.*

Proof. Consider the canonical $\mathcal{A}$-group $W = W_1 \times \cdots \times W_k$ where $W_i = W_{r_i,s_i} = A_{r_i} \mathrm{wr} A_{s_i}$, and let H be a finitely generated subgroup of W. We need to check that H satisfies conditions (i)-(iv) in the definition of an $\mathcal{A}$-group. This is obvious for (i),(ii) and (iv), so it remains to consider condition (iii). Since the result is obviously true if H is abelian, we assume that H is not abelian. Let H_i $(i = 1, \dots, k)$ denote the projection of H onto W_i. Since subgroups of ρ-groups are themselves ρ-groups, Proposition 5.1 gives that the H_i are $\mathcal{A}$-groups, and then Proposition 5.2 tells us that $\overline{H} = H_1 \times \ldots \times H_k$ is an $\mathcal{A}$-group. Let $\overline{M} = \mathrm{Fit}(\overline{H})$ and $\overline{A} = \overline{H}/\overline{M}$. Then $\overline{M}$ is an $\mathcal{A}$-modules over $\mathbb{Z}\overline{A}$ (since $\overline{H}$ is an $\mathcal{A}$-group). We claim that $\mathrm{Fit}(H) = \mathrm{Fit}(\overline{H}) \cap H$. To verify the claim we need to show that $\mathrm{Fit}(H) \subseteq \mathrm{Fit}(\overline{H}) \cap H$ (the inverse inclusion is obvious). Suppose that $h \in \mathrm{Fit}(H)$. Write $h = (h_1, \dots, h_k)$ with $h_i \in H_i$, and let $\varphi_i : H \to H_i$ denote the projection of H onto H_i $(i = 1, \dots, k)$. Since φ_i is a surjective homomorphism, $(\mathrm{Fit}(H)\varphi_i) \subseteq \mathrm{Fit}(H_i)$. Hence $h_i \in \mathrm{Fit}(H_i)$ $(i = 1, \dots, k)$ and, consequently, $h \in \mathrm{Fit}(\overline{H})$. This proves our claim that $\mathrm{Fit}(H) = \mathrm{Fit}(\overline{H}) \cap H$. But then the quotient $A = H/\mathrm{Fit}(H)$ can be identified with a subgroup of $\overline{A} = \overline{H}/\overline{M}$, and now Proposition 4.2 tells us that $\mathrm{Fit}(H)$ is an $\mathcal{A}$-module. □

The main result of this section is that the converse of Proposition 5.3 is also true.

Theorem 5.1. *Any $\mathcal{A}$-group G is isomorphic to a subgroup of a suitable canonical group W.*

The proof of this theorem will take up most of the rest of this section, and requires a number of lemmas. In the first of these lemmas we derive a free presentation for an $\mathcal{A}$-group. To this end, we now introduce some notation that will also be used in the next section. Consider the free abelian group A_n of rank n with free generators $a_1, \dots, a_n$, and X be an alphabet that includes the letters $x_1, \dots, x_n$. Let α be an arbitrary non-zero element of the integral group ring $\mathbb{Z}A_n$ with $\mathrm{supp}(\alpha) = \{g_1, \dots, g_s\}$. Then

$$\alpha = \sum_{i=1}^{s} n_i g_i$$

where the n_i are non-zero integer coefficients, and each of the elements $g_i \in A_n$ has a unique expression

$$g_i = a_1^{\lambda_{i1}} a_2^{\lambda_{i2}} \cdots a_n^{\lambda_{in}} \quad (\lambda_{i1}, \dots, \lambda_{in} \in \mathbb{Z})$$

in terms of the free generators of A_n. For any $\alpha \in \mathbb{Z}A_n$ as above, and any group word w in the alphabet X we define a group word w^α in the alphabet X by setting

$$w^{\alpha} = (w^{n_1})^{x_1^{\lambda_{11}} x_2^{\lambda_{12}} \cdots x_n^{\lambda_{1n}}} (w^{n_2})^{x_1^{\lambda_{21}} x_2^{\lambda_{22}} \cdots x_n^{\lambda_{2n}}} \cdots (w^{n_s})^{x_1^{\lambda_{s1}} x_2^{\lambda_{s2}} \cdots x_n^{\lambda_{sn}}}. \tag{5.1}$$

Now let G be an $\mathcal{A}$-group. If G is abelian, it has to be free abelian, and hence it is itself a canonical $\mathcal{A}$-group. We therefore assume from now on that G is not abelian. Let $M = \mathrm{Fit}(G)$, then G/M is a free abelian group, $G/M \cong A_n$ say. Let $a_1, \ldots, a_n$ be a system of free generators for A_n, $x_1, \ldots, x_n$ their inverse images in G, and let $y_1, \ldots, y_m \in M$ be a system of generators for M as a $\mathbb{Z}A_n$-module. We will use multiplicative notation for the module M. Since $G' \leq M$, we have that, for all pairs x_i, x_j with $1 \leq i < j \leq n$, the commutator $[x_i, x_j]$ can be written as

$$[x_i, x_j] = y_1^{\alpha_{ij1}} y_2^{\alpha_{ij2}} \cdots y_m^{\alpha_{ijm}} \tag{5.2}$$

for some $\alpha_{ijk} \in \mathbb{Z}A_n$. Furthermore, let

$$M = \langle y_1, \ldots, y_m \mid r_1 = 1, \ldots, r_t = 1 \rangle$$

be a free presentation of M. Here each r_i $(i = 1, \ldots, q)$ is an element of the free $\mathbb{Z}A_n$-module on $y_1, \ldots, y_m$, and hence each of the r_i has a unique expression as

$$r_i = y_1^{\beta_{i1}} y_2^{\beta_{i2}} \cdots y_m^{\beta_{im}} \tag{5.3}$$

for some $\beta_{ik} \in \mathbb{Z}A_n$. In the following lemma, the relations in (5.4) involve group words, which have to be read in accordance with the definition in (5.1).

Lemma 5.1. *The group G admits a free presentation of the form*

$$\begin{aligned} G = \langle x_1, \ldots, x_n,\ y_1, \ldots, y_m \mid & \\ & [x_i, x_j] = y_1^{\alpha_{ij1}} y_2^{\alpha_{ij2}} \cdots y_m^{\alpha_{ijm}} \qquad (1 \leq i < j \leq n), \\ & y_1^{\beta_{i1}} y_2^{\beta_{i2}} \cdots y_m^{\beta_{im}} = 1 \qquad (i = 1, \ldots, t), \\ & [y_i^{\alpha}, y_j^{\beta}] = 1 \qquad (1 \leq i, j \leq m,\ \alpha, \beta \in \mathbb{Z}A_n) \rangle, \end{aligned} \tag{5.4}$$

where the α_{ijk} and the β_{ik} are as in (5.2) *and* (5.3), *respectively.*

Proof. Let G_1 denote the group given by the free presentation (5.4). It is clear that there is a surjective homomorphism $\varphi : G_1 \to G$ with $\ker \varphi \subseteq M_1$ where M_1 denotes the normal closure of $y_1, \ldots, y_s$ in G_1. Moreover, $M_1 \varphi = M$. Since there is an inverse homomorphism $\phi : M \to M_1$, we have $\ker \varphi = \{1\}$. $\square$

Since G is an $\mathcal{A}$-group, M is an $\mathcal{A}$-module over $\mathbb{Z}A_n$. Let $\{1\} = \bigcap_{s=1}^{l} Q_s$ be an $\mathcal{A}$-primary decomposition for the trivial submodule $\{1\}$ of M, i.e. the modules $M_s = M/Q_s \quad (s = 1, \ldots, l)$ are primary $\mathcal{A}$-modules, and let $\mathrm{Ass}(M_s) = \{\Delta_{B_s}\}$ with $B_s \in \mathcal{P}(A_n)$.

Lemma 5.2. *With the notation introduced above, the centre $Z(G)$ is non-trivial if and only if there exists an $s_0 \quad (1 \leq s_0 \leq l)$ such that $B_{s_0} = A_n$.*

Proof. Since G is a $\mathcal{A}$-group, it is easily verified that the centre of G is precisely $M[\Delta_{A_n}]$, and the lemma follows. □

Now let $\pi_s \quad (1 \leq s \leq l)$ denote the natural projection of M onto M_s. Then

$$\pi : M \to M_1 \times \cdots \times M_l \quad u \mapsto (u\pi_1, \ldots, u\pi_l)$$

$(u \in M)$ is an injective homomorphism of $\mathbb{Z}A_n$-modules. Since the M_s are finitely generated torsion-free $\mathbb{Z}(A_n/B_s)$-modules, we can find injective homomorphisms $\nu_s : M_s \to T_s$ where the T_s are finitely generated free $\mathbb{Z}(A_n/B_s)$-modules. This is a well-known fact (a typical instance of folklore) from commutative algebra, which can easily be established by working in the vector space obtained by tensoring M_s with the field of fractions of the domain $\mathbb{Z}(A_n/B_s)$. It is clear that the T_s may be regarded as a $\mathbb{Z}A_n$-module (with B_s acting trivially) , and we will adopt this point of view. Then ν_s is a homomorphism of $\mathbb{Z}A_n$-modules. We write φ_s for the composite of π_s and ν_s:

$$\varphi_s : M \to T_s \qquad u \mapsto u\pi_s\nu_s$$

Let $\widetilde{W_s}$ denote the semidirect product of T_s and A. The groups $\widetilde{W_i}$ are not necessarily wreath products, but they are in a certain sense close to being wreath products. Let $\widetilde{W} = \widetilde{W_1} \times \cdots \times \widetilde{W_l}$. Our next goal is to derive an embedding of G into $\widetilde{W}$. We set

$$\delta = a_1 + \cdots + a_n - n \in \mathbb{Z}(A_n).$$

First assume that $Z(G) = 1$. We start by constructing an embedding ψ of M into the module $T_1 \times \cdots \times T_l \subseteq \widetilde{W}$. For each s with $1 \leq s \leq l$ we define a homomorphism $\psi_s : M \to T_i$ by setting

$$u\psi_s = (u\varphi_s)^{\delta}$$

for all $u \in M$, and then we put

$$u\psi = (u\psi_1, \ldots, u\psi_l).$$

Lemma 5.3. *The homomorphism ψ is an embedding of M into $T_1 \times \ldots \times T_l$.*

Proof. If $u \in \ker\psi$ then $(u\varphi_s)^\delta = 1$ for all s $(1 \leq s \leq l)$, and hence $(u\pi_s)^\delta = 1$ for all s. But since $B_s \neq A_n$, it follows that $\delta \notin \Delta_{B_s}$. Thus $(u\pi_s)^\delta = 1$ implies $u\pi_s = 1$, and it follows that $\ker\psi = \{1\}$. □

Now we extend the embedding ψ to an embedding of G into $\widetilde{W}$. For each s with $1 \leq s \leq l$ and each i with $1 \leq i \leq n$ we set

$$x_i\theta_s = (a_i, (\prod_{k=1}^{n}[x_i, x_k])\varphi_s) \in \widetilde{W}_s,$$

and for all $u \in M$ we set

$$u\theta_s = (1, u\psi_s) \in \widetilde{W}_s.$$

Lemma 5.4. *The map θ_s extends to a homomorphism of G into $\widetilde{W}_s$ (which will also be denoted by θ_s).*

Proof. We need to verify that θ_s respects the defining relations (5.4). This is obvious for the relations involving only $y_1, \ldots, y_m$, and so it remains to check the relations involving the commutators $[x_i, x_j]$. Using standard commutator identities we find

$$\begin{aligned}
&(x_i\theta_s)^{-1}(x_j\theta_s)^{-1}x_i\theta_s x_j\theta_s \\
&= (1, (\prod_{k=1}^{n}[x_j, x_k])\varphi_s^{(1-a_i)}(\prod_{k=1}^{n}[x_i, x_k])\varphi_s^{(a_j-1)}) \\
&= (1, ((\prod_{k=1}^{n}[x_j, x_k]^{(1-a_i)})(\prod_{k=1}^{n}[x_i, x_k])^{(a_j-1)})\varphi_s) \\
&= (1, (\prod_{k=1}^{n}[x_j, x_k, x_i][x_k, x_i, x_j])\varphi_s) \\
&= (1, (\prod_{k=1}^{n}[x_j, x_i, x_k]\varphi_s) \\
&= (1, [x_i, x_j]^{\sum_{k=1}^{n}(x_k-1)}\varphi_s) \\
&= (1, [x_i, x_j]^\delta\varphi_s) \\
&= (1, [x_i, x_j]\psi_s) \\
&= (y_1\theta_s)^{\beta_{i1}}(y_2\theta_s)^{\beta_{i2}} \cdots (y_m\theta_s)^{\beta_{im}}
\end{aligned}$$

as required. □

Using the homomorphisms θ_s we now define a homomorphism

$$\theta : G \to \widetilde{W}$$

by setting

$$g\theta = (g\theta_1, \dots, g\theta_l) \in \widetilde{W_1} \times \cdots \times \widetilde{W_l} = \widetilde{W}$$

for all $g \in G$.

Lemma 5.5. *The homomorphism θ is an embedding of G into $\widetilde{W}$.*

Proof. Since the restriction of θ to $M = \mathrm{Fit}(G)$ coincides with ψ, this follows by Lemma 5.3. □

Now we consider the case where $Z(G) \neq 1$. Then we have $Z(G) = M[\Delta_{A_n}]$ by Lemma 5.2, and $\Delta_{A_n} \in \mathrm{Ass}(G)$, say $\Delta_{A_n} = \Delta_{B_1}$. Then $M[\Delta_{A_n}] = \bigcap_{s=2}^{l} Q_s$ by Lemma 4.2. In this case we construct Q_1 in the following special way. We put $Q_1 = \mathrm{Is}(G')$, where $\mathrm{Is}(G')$ is the isolator of G' in G . Since $G' \cap Z(G) = 1$ in G, we have $\mathrm{Is}(G') \cap Z(G) = \{1\}$ in M. Now, if we put $Q_1 = \mathrm{Is}(G')$, then $\bigcap_{i=1}^{l} Q_i = Q_1 \cap Z(G) = \{1\}$. Moreover $G/\,\mathrm{Is}(G')$ is a torsion-free abelian group. Consequently, $\{1\} = \bigcap_{i=1}^{l} Q_i$ is an $\mathcal{A}$-primary decomposition for the trivial submodule $\{1\}$ of M. Now we define an embedding $\theta : G \to \widetilde{W}$ similar to the centre-free case, but with one essential modification. With φ_s, ψ_s $(s = 1, \dots, l)$ as above we leave $\theta_2, \dots, \theta_l$ the same as before, but we now define θ_1 by setting

$$x_i\theta_1 = (a_i, (\prod_{k=1}^{n}[x_i, x_k])\varphi_1) \in \widetilde{W_1}$$

(that is still exactly as in the centre-free case), and for all $u \in M$ we set

$$u\theta_1 = (1, u\varphi_1) \in \widetilde{W_1}.$$

We claim that θ_1 extends to a homomorphism from G to $\widetilde{W_1}$, and as in the proof of Lemma 5.4 we need to check that θ_1 respects the defining relations (5.4). By repeating the calculations from the proof of Lemma 5.4 line by line, we find that

$$(x_i\theta_1)^{-1}(x_j\theta_1)^{-1}x_i\theta_1 x_j\theta_1 = (1, [x_i, x_j]^{\delta}\varphi_1)$$

Since G' is contained in Q_1, we have $[x_i, x_j]\varphi_1 = 1$, and hence

$$(1, [x_i, x_j]^{\delta}\varphi_1) = (1, [x_i, x_j]\varphi_1) \ \ (= 1),$$

which is all we need for the proof of Lemma 5.4 to go through. Thus we have again a homomorphism $\theta : G \to \widetilde{W}$. This time the restriction to M is given by

$$u\theta|_M = (u\varphi_1, u\psi_2, \dots, u\psi_l) \in T_1 \times T_2 \times \dots T_l,$$

and an argument similar to that of Lemma 5.3 shows that this is injective. Hence so is θ, and thus we have got the required embedding of G into $\widetilde{W}$.

Finally, consider the groups $\widetilde{W}_s$ $(s = 1, \dots, l)$. Recall that $\widetilde{W}_s$ is the semidirect product of A_n and and a finitely generated free $\mathbb{Z}(A_n/B_s)$-module, which was regarded as a $\mathbb{Z}(A_n)$-module with B_s acting trivially. Hence the subgroup B_s is central in $\widetilde{W}_s$ and the quotient $W_s = \widetilde{W}_s/B_s$ is the semidirect product of A_n/B_s and a finitely generated free $\mathbb{Z}(A_n/B_s)$-module. In other words, W_s is a wreath product of two free abelian groups of finite rank. Put $W = W_1 \times \dots \times W_l$. Let $\xi_s : \widetilde{W}_s \to W_s$ denote the natural homomorphism from $\widetilde{W}_s$ onto W_s, and let $\xi : \widetilde{W} \to W$ be the induced homomorphism from $\widetilde{W}$ onto W. It is easily seen that $G\theta \cap \ker \xi = \{1\}$. Indeed, since $\ker \xi$ is a central subgroup, $G\theta \cap \ker \xi \subseteq Z(G\theta)$. If $Z(G) = 1$, the result is clear. If $Z(G) \neq 1$, then $\ker \xi \cap \theta(G) \subseteq \widetilde{W}_1 = A \oplus T_1$, where T_1 is a free abelian group of finite rank. Since $\ker \xi_1 = A$ and $M[\Delta_{A_n}] \subseteq T_1$, it follows that $G\theta \cap \ker \xi = 1$. Hence $\theta\xi$ is an embedding of G into W. This completes the proof of Theorem 5.1. □

The key technical result of [3] is a special case of Theorem 5.1.

Corollary 5.1. ([3, Lemma 3.8]) *Any finitely generated ρ-group is isomorphic to a subgroup of a wreath product $W_{r,s}$ for some suitable natural numbers r and s.*

Proof. Indeed, if G is non-abelian and $M = \mathrm{Fit}(G)$, then $\mathrm{Ass}(M) = \{0\}$, and the corollary follows from the proof of Theorem 5.1. If G is abelian, the assertion is obvious. □

Another consequence of our embedding theorem is the following.

Corollary 5.2. *Any finitely generated subgroup of an $\mathcal{A}$-group is itself an $\mathcal{A}$-group.*

Proof. This follows immediately from Theorem 5.1 and Proposition 5.3. □

6. Defining quasi-identities for the quasivariety generated by a free metabelian group

In this Section we exhibit a system of quasi-identities that, as we will prove later, determines the quasivariety generated by a non-cyclic free metabelian group. Our system consists of six sets of quasi-identities. The first four of them are straightforward.

The first set consists of a single identity,

$$\forall\, x, y, z, t\ ([[x, y], [z, t]] = 1), \tag{6.1}$$

which determines the variety of all metabelian groups.

The second set is a system of quasi-identities defining the quasivariety of all torsion-free groups:

$$\forall\, x\ (x^n = 1 \Longrightarrow x = 1) \quad (n \in \mathbb{N}) \tag{6.2}$$

The third set consists again of a single quasi-identity that determines the quasivariety of all groups in which every nilpotent subgroup is abelian:

$$\forall\, x, y\ ([x, y, x] = 1 \wedge [x, y, y] = 1 \Longrightarrow [x, y] = 1) \tag{6.3}$$

The fourth set is a system of quasi-identities that, if it holds for some torsion-free metabelian group G, guarantees that the Fitting subgroup $\mathrm{Fit}(G)$ is isolated in G:

$$\forall\, x, y\ ([x^n, (x^n)^y] = 1 \Longrightarrow [x, x^y] = 1) \quad (n \in \mathbb{N}) \tag{6.4}$$

The fifth set of quasi-identities requires some preliminary discussion. For any natural number n, consider the free abelian group A_n with free generators $a_1, \ldots, a_n$ and the integral group ring $R_n = \mathbb{Z}A_n$. Let α be a non-zero element of the augmentation ideal Δ_{A_n}, and let $C(\alpha)$ be the contents of α as defined in Section 2.2. Using the contents $C(\alpha)$ we now define $\det(\alpha)$, the determinant of α, by setting

$$\det(\alpha) = \prod_{a \in C(\alpha)} (1 - a).$$

Recall that $\mathrm{Rad}(\alpha)$ denotes the radical of α, that is the smallest radical ideal of R containing α. We know that $\mathrm{Rad}(\alpha)$ is a finitely generated ideal in R_n. Let $\beta_1, \ldots, \beta_q$ be generators for that ideal:

$$\mathrm{Rad}(\alpha) = \langle \beta_1, \ldots, \beta_q \rangle\,.$$

Note that, by Proposition 3.2, there is an effective algorithm for finding $\beta_1, \ldots, \beta_q$ for any given α. Finally, recall the definition (5.1) of w^α, where w is a word in the language of group theory and $\alpha \in R_n$.

Now we are ready to introduce the fifth group of quasi-identities. It consists of an infinite series of sets of quasi-identities, one for each natural number n, which refers to R_n, and each of these sets consists in its turn of three infinite subsets as follows.

Firstly, for all non-zero $\alpha \in R_n$ with $\varepsilon(\alpha) \neq 0$, we write

$$\forall\, y, z, x_1, \ldots, x_n \; ([y, z]^\alpha = 1 \Longrightarrow [y, z] = 1). \tag{6.5}$$

Secondly, for all non-zero $\alpha \in R_n$ with $\varepsilon(\alpha) = 0$, we write

$$\forall\, y, z, x_1, \ldots, x_n \; ([y, z]^\alpha = 1 \Longrightarrow [y, z]^{\det(\alpha)} = 1). \tag{6.6}$$

Thirdly, for all non-zero $\alpha \in R_n$ with $\varepsilon(\alpha) = 0$ and $\mathrm{Rad}(\alpha) = \langle \beta_1, \ldots, \beta_q \rangle$, we write

$$\forall\, y, z, x_1, \ldots, x_n \; ([y, z]^\alpha = 1 \Longrightarrow [y, z]^{\beta_i} = 1) \tag{6.7}$$

where $i = 1, 2, \ldots, q$.

The sixth and last set of quasi-identities is again more straightforward. It guarantees that for any finitely generated subgroup, centre and commutator subgroup have trivial intersection. For any natural number n, we write

(6.8)

$$\forall\, x_1, \ldots, x_n, y_1, \ldots, y_n, z \; (z = \prod_{i=1}^{n} [x_i, y_i] \wedge \bigwedge_{i=1}^{n} ([z, x_i] = [z, y_i] = 1 \Rightarrow z = 1).$$

Let $\mathcal{Q}$ denote the system consisting of all the quasi-identities (6.1)-(6.8).

Definition. A group is called a q-group if it satisfies all quasi-identities of the system $\mathcal{Q}$.

Notice that the system $\mathcal{Q}$ is recursive. This is obvious for (6.1)-(6.6) and for (6.8), and for (6.7) it follows from Proposition 3.2. In the final section of this paper we will prove that that the class of all q-groups coincides with the quasivariety $\mathrm{qvar}(F)$. In other words, $\mathcal{Q}$ is a defining system of quasi-identities for this quasivariety (see Corollary 7.1).

Before we go any further, we wish to point out that one set of axioms in $\mathcal{Q}$ is redundant in that it is a consequence of the others.

Remark 6.1. The quasi-identities (6.4) are consequences of the quasi-identities (6.1)-(6.3), (6.5) and (6.6) in $\mathcal{Q}$.

Proof. Let G be a group satisfying the quasi-identities (6.1)-(6.3) and (6.5)-(6.6), and suppose that $[g^n, (g^n)^h] = 1$ for some $g, h \in G$ and some natural number n. We need to show that $[g, g^h] = 1$. Using standard commutator identities we obtain

$$[g^n, (g^n)^h] = [g^n, g^n[g^n, h]] = [g^n, h]^{1-\overline{g}^n}$$

where $\overline{g}$ is the image of g in the factor group $G/\operatorname{Fit}(G)$. Now observe that $\det(1-x^n) = 1-b$ where b is the highest possible root of x. Since $[g^n, h]^{1-\overline{g}^n} = 1$, the quasi-identity (6.6) gives that $[g^n, h]^{1-\overline{a}} = 1$ where $\overline{g} \in \langle \overline{a} \rangle$. But this of course implies that $[g^n, h]^{1-\overline{g}} = 1$. Since

$$[g^n, h] = [g, h]^{1+\overline{g}+\cdots+\overline{g}^{n-1}},$$

we have, in fact, that

$$[g, h]^{(1+\overline{g}+\cdots+\overline{g}^{n-1})(1-\overline{g})} = 1,$$

so

$$[g, h]^{(1-\overline{g})(1+\overline{g}+\cdots+\overline{g}^{n-1})} = [g, [g, h]]^{1+\overline{g}+\cdots+\overline{g}^{n-1}} = 1$$

Since $\varepsilon(1+x+\cdots+x^{n-1}) = n \neq 0$, the quasi-identity (6.5) now implies that $[g, [g, h]] = 1$. But

$$[g, [g, h]] = [g, g[g, h]] = [g, g^h].$$

Hence $[g, g^h] = 1$ as required. □

In view of the above remark, the quasi-identities (6.4) could be deleted from the system $\mathcal{Q}$, but for technical reasons (see, e.g., Lemma 6.1) we find it convenient to keep it in there.

Lemma 6.1. *Let G be a group satisfying the quasi-identities* (6.1)-(6.4)*. Then*

(i) *G is a torsion-free metabelian group,*
(ii) *the Fitting subgroup* $\operatorname{Fit}(G)$ *is abelian,*
(iii) *the quotient $G/\operatorname{Fit}(G)$ is a torsion-free abelian group.*

Proof. Any group satisfying (6.1) and (6.2) is plainly metabelian and torsion-free, whence (i). Moreover, for any group G, the Fitting subgroup $\operatorname{Fit}(G)$ is locally nilpotent. If $\operatorname{Fit}(G)$ is not abelian, it contains a 2-generator non-abelian nilpotent group of class two, which is, however, impossible if G satisfies (6.3). This gives (ii). For any metabelian group G, $\mathrm{Fit}(G)$ contains the commutator subgroup G', and hence the quotient $G/\mathrm{Fit}(G)$ is abelian. In order to show that this quotient is also torsion-free, suppose there exists an element $x \in G$ such that $x^l \in \mathrm{Fit}(G)$. Then we have that $(x^l)^y \in \mathrm{Fit}(G)$ for all $y \in G$. Since $\mathrm{Fit}(G)$ is abelian, we get that $[x^l, (x^l)^y] = 1$, and then (6.4) implies that $[x, x^y] = 1$. The latter gives $[x, y, x] = 1$ for all $y \in G$. For $y \in \mathrm{Fit}(G)$ we also have $[x, y, y] = 1$, and then (6.3) yields that $[x, y] = 1$ for all $y \in \mathrm{Fit}(G)$. But then $\langle \mathrm{Fit}(G), x\rangle$ is an abelian normal subgroup of G, which finally gives $x \in \mathrm{Fit}(G)$. Hence $G/\mathrm{Fit}(G)$ is torsion-free, and this completes the proof of the lemma. $\square$

Lemma 6.2. *Any finitely generated q-group is an $\mathcal{A}$-group.*

Proof. Let G be a finitely generated q-group. In view of Lemma 6.1 and the fact that $Z(H) \cap H' = 1$ for any finitely generated subgroup H in G, it is sufficient to show that $\mathrm{Fit}(G)$ is an $\mathcal{A}$-module over the integral group ring of $G/\mathrm{Fit}(G)$. We may assume that G is non-abelian that is $\mathrm{Fit}(G) \geq G' \neq 1$. Let $G/\mathrm{Fit}(G) \cong A_n$ and $R = \mathbb{Z}A_n$. We first consider a non-trivial element of $\mathrm{Fit}(G)$ that is a single commutator. Let $u = [y, z] \in \mathrm{Fit}(G)$ with $u \neq 1$ and $y, z \in G$, and suppose that $\mathrm{Ann}(u) \neq \{0\}$. Let $\alpha \in \mathrm{Ann}(u)$ with $\alpha \neq 0$. Then the quasi-identity (6.7) implies that $\mathrm{Rad}(\alpha) \leq \mathrm{Ann}(u)$. If $\mathrm{Ann}(u) = \mathrm{Rad}(\alpha)$, the annihilator has the required form (see Remark 4.1). Suppose then that this is not the case, and let $\beta \in \mathrm{Ann}(u) \setminus \mathrm{Rad}(\alpha)$. Then, by Corollary 3.1 and the proof of Proposition 3.1, there exists an element γ such that $\mathrm{Rad}(\alpha, \beta) = \mathrm{Rad}(\gamma)$ and $\gamma = \alpha - \beta^k$ where k is a sufficiently large positive integer. Then $\gamma \in \mathrm{Ann}(u)$, and hence $\mathrm{Rad}(\gamma) \leq \mathrm{Ann}(u)$. If $\mathrm{Rad}(\gamma) \neq \mathrm{Ann}(u)$, we may repeat the above procedure to produce a chain of ideals

$$\mathrm{Rad}(\alpha) \leq \mathrm{Rad}(\gamma) \leq \mathrm{Rad}(\gamma_1) \leq \cdots \leq \mathrm{Ann}(u).$$

Since R is Noetherian, we eventually get that $\mathrm{Ann}(u)$ coincides with the radical of one of its elements, and hence the annihilator is of the required form.

Now let $u \in \mathrm{Fit}(G)$ be an arbitrary non-trivial element. Suppose that the free abelian group A_n is freely generated by $a_1, \ldots, a_n$ and let $x_1, \ldots, x_n \in G$ be inverse images of these free generators in G. Then $G = \langle x_1, \ldots, x_n, \mathrm{Fit}(G)\rangle$. We distinguish two cases. First assume that $[u, x_i] = 1$ for $i = 1, \ldots, n$. Then $u \in Z(G)$, and hence $\mathrm{Ann}(u) = \Delta_{A_n}$.

If that is not the case, there exists an x_i such that $[u, x_i] \neq 1$. Let u_i denote the commutator $u_i = [u, x_i]$ $(i = 1, \ldots, n)$. If $\alpha \in \mathrm{Ann}(u)$, then $\alpha \in \mathrm{Ann}(u_i)$ $(i = 1, \ldots, n)$. Consequently, $\mathrm{Ann}(u) \subseteq \bigcap_{i=1}^{n} \mathrm{Ann}(u_i)$. We have that

$$\bigcap_{i=1}^{n} \mathrm{Ann}(u_i) = (\Delta_{A_n} : \mathrm{Ann}(u)) \tag{6.9}$$

Assume that $\mathrm{Ann}(u) \neq \bigcap_{i=1}^{n} \mathrm{Ann}(u_i)$, and let $\beta \in \bigcap_{i=1}^{n} \mathrm{Ann}(u_i) \setminus \mathrm{Ann}(u)$. Then, since at least one of the elements u_i is non-zero, we have that $\beta \in \Delta_{A_n}$ and $u^\beta \neq 1$. Since $\beta \in \Delta_{A_n}$, $u^\beta \in G'$. On the other hand, in view of (6.9), $u^{\beta(x_i - 1)} = 1$ for all $i = 1, \ldots, n$, and, consequently, $u^\beta \in Z(G)$. This gives $u^\beta \in Z(G) \cap G'$ and so $u^\beta = 1$, that is $\beta \in \mathrm{Ann}(u)$, which contradicts our assumptions. Hence $\mathrm{Ann}(u) = \bigcap_{i=1}^{n} \mathrm{Ann}(u_i)$, and, consequently, it has the required form. □

Lemma 6.3. *The free metabelian group F is a q-group.*

Proof. Let F be a free metabelian group of rank $m > 1$. Then $\mathrm{Fit}(G) = F'$, $F/\mathrm{Fit}(F) \cong A_m$, and $Z(F) = 1$. It is clear that F satisfies the quasi-identities (6.1)-(6.4). If H is a non-abelian subgroup of F, then $Z(H) = 1$ since H satisfies the axiom CT. Consequently (6.8) holds for F.

It remains to check the axioms (6.5)-(6.7). First of all, F satisfies the quasi-identities (6.5) since F' is a torsion-free module for $\mathbb{Z}(F/F')$ (see Section 2.1).

Now suppose that (6.6) is violated on the elements $y = h_1, z = h_2, x_1 = f_1, \ldots, x_n = f_n$ of F. Then $[h_1, h_2] \neq 1$. Recall that the α in (6.6) is an element of the augmentation ideal of the group ring R_n of the free abelian group A_n on free generators $a_1, \ldots, a_n$ for some n, say $\alpha = \sum \lambda_i g_i \in \Delta_{A_n}$ with $g_i = a_1^{\delta_{i1}} a_2^{\delta_{i2}} \ldots a_n^{\delta_{in}}$. We write $\overline{f}_1, \ldots, \overline{f}_n$ for the natural images of $f_1, \ldots f_n$ in F/F', and $\alpha(\overline{f})$ for the element of $\mathbb{Z}(F/F')$ obtained from α by the substitution $a_i \mapsto \overline{f}_i$. Since F' is a torsion-free $\mathbb{Z}(F/F')$-module, $\alpha(\overline{f}) = 0$. Consequently, there is at least one pair of indices (j_1, j_2) $j_1 \neq j_2$ such that λ_{j_1} and λ_{j_2} have opposite signs and $g_{j_1}(\overline{f}) \equiv g_{j_1}(\overline{f}) \mod F'$. Furthermore, the root b of the element $g_{j_1} g_{j_2}^{-1}$ is, by definition, contained in $C(\alpha)$ (see Section 3). Hence $1 - b$ divides $\det(\alpha)$. But $[h_1, h_2]^{b(\overline{f})-1} = 1$ in F, contradicting our assumptions.

Finally, assume that (6.7) is violated on the elements $y = h_1, z = h_2, x_1 = f_1, \ldots, x_n = f_n$ of F. Then $[h_1, h_2] \neq 1$ and $\alpha(\overline{f}) = 0$. Let ϕ be the homomorphism from A_n to F/F' such that $a_i\phi = \overline{f}_i$ and let B denote the kernel of ϕ. Then $\alpha \in \Delta_B$. Consequently (see Section 3), the $\beta_i \ \ (i = 1, \ldots, q)$ are also contained in Δ_B, and hence $[h_1, h_2]^{\beta_i(\overline{f})} = 1$ in F. This completes the proof of the lemma. □

7. The main results

In this Section F denotes a non-cyclic free metabelian group of finite rank. Let $\mathcal{U}$ denote the system of universal sentences consisting of the quasi-identities (6.1), (6.2), (6.5) and (6.6) from Section 6, and the axiom CT (see Section 2.1).

Definition. A group is called a u-group if it satisfies all axioms of the system $\mathcal{U}$.

First of all we note that F is a u-group. It satisfies the quasi-identities in $\mathcal{U}$ by Lemma 6.3, and the fact that CT holds in F is an immediate consequence of Malcev's result on centralizers in F (see Section 2.1). We attribute our first result to O. Chapuis, who proved a similar Theorem in his paper [3]. However, Theorem A below is a substantial modification of Chapuis' original result as we use a different system of universal sentences. We also feel that our proof, in particular in its main ingredient Corollary 5.1 (the counterpart of Lemma 3.8 in [3]) is more direct and technically less involved than the original proof in [3].

Theorem A . (O. Chapuis) *For a finitely generated group G, the following statements are equivalent.*

(i) *G is a subgroup of a wreath product $W_{r,s}$ for some positive integers r, s.*

(ii) $G \in \mathrm{ucl}(F)$.

(iii) *G is a u-group.*

(iv) *G is a ρ-group.*

Proof. For the implication (i)$\Longrightarrow$(ii) we refer to Chapuis' paper [3, Lemma 3.1], where he proves that $W_{r,s} \in \mathrm{ucl}(F)$. The implication (ii)$\Longrightarrow$(iii) is an immediate consequence of the above mentioned fact that F is a u-group, which gives that any $G \in \mathrm{ucl}(F)$ is a u-group. Now we turn to the implication (iii)$\Longrightarrow$(iv). Clearly, the conditions (i) and (ii) in the definition of a ρ-group hold for any u-group. It therefore remains to show that condition (iii) holds as well that is, for any u-group G, $\mathrm{Fit}(G)$ is a torsion-free module over the integral group ring of $G/\mathrm{Fit}(G)$. Let $A_n = G/\mathrm{Fit}(G)$ and let $R_n = \mathbb{Z}A_n$. If G is abelian there is nothing to prove, so assume that $G' \neq 1$. Let $f, g \in G$ such that $[f, g] \neq 1$. We will show that $\mathrm{Ann}([f, g]) = \{0\}$. Let $\alpha \in \mathbb{Z}A_n$ with $\alpha \neq 0$ such that $[f, g]^\alpha = 1$. Then (6.5) gives that $\varepsilon(\alpha) = 0$, and then (6.6) implies that $[f, g]^{\det(\alpha)} = 1$. This gives, in turn, that there exists an $u \in G'$ ($u \neq 1$) such that $u^{(1-b)} = 1$, where $b \in C(\alpha)$. Let g be an inverse image of b under the natural homomorphism $G \to A_n$. Then $u^{(1-b)} = 1$ gives $[u, g] = 1$. But we also have $[u, v] = 1$ for all $v \in \mathrm{Fit}(G)$. Now the axiom CT yields that g commutes with all elements in $\mathrm{Fit}(G)$. But then $g \in \mathrm{Fit}(G)$, which is, by construction, impossible. Now let $u \in \mathrm{Fit}(G)$ be an arbitrary non-trivial element. Let $x_1, \dots, x_n \in G$ as in the proof of Lemma 6.2. If $[u, x_i] = 1$ for all x_i, then u is central, and CT implies that G is abelian, contradicting our original assumption on G. Hence there exists an x_i such that $[u, x_i] \neq 1$. Now if $u^\alpha = 1$, then $u^{(1-a_i)\alpha} = 1$, which is the same as to say that $[u, x_i]^\alpha = 1$. But now α annihilates a single commutator, and we have already shown that this implies $\alpha = 0$. This establishes the implication (iii)$\Longrightarrow$(iv). For the final implication (iv)$\Longrightarrow$(i) we refer to Corollary 5.1.

□

Now we are ready for the main result of this paper.

Theorem B. *For a finitely generated group G, the following statements are equivalent.*

(i) *G is a subgroup of a direct product $W_{r_1,s_1} \times \ldots \times W_{r_k,s_k}$ for some positive integers $r_1, s_1, \dots, r_k, s_k$,*

(ii) $G \in \mathrm{qvar}(F)$.

(iii) *G is a q-group.*

(iv) *G is an $\mathcal{A}$-group.*

Proof. By Theorem A, $W_{r,s} \in \mathrm{ucl}(F)$, and since $\mathrm{ucl}(F) \subseteq \mathrm{qvar}(F)$ and $\mathrm{qvar}(F)$ is closed under taking subgroups and finite direct products, this yieds the implication (i)$\Longrightarrow$(ii). The implication (ii)$\Longrightarrow$(iii) holds by Lemma 6.3 which tells us that F, and hence any group in $\mathrm{qvar}(F)$, is a q-group. Lemma 6.2, which says that any finitely generated q-group is an $\mathcal{A}$-group, gives the implication (iii)$\Longrightarrow$(iv). Finally, the implication (iv)$\Longrightarrow$(i) has been established in Theorem 5.1. This completes the proof of Theorem B. □

Our concluding result is an immediate consequence of Theorem B.

Corollary 7.1. *The system $\mathcal{Q}$ is a defining system of quasi-identities for the quasivariety* $\mathrm{qvar}(F)$.

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